Fondamenti di Segnali e Trasmissione

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1 | I segnali

Signals represent the behavior of physical magnitudes as a function of one or more independent variables.

Signals can be **continuous** $x(t)$ or **discrete** x_n, **real** or **complex**, **deterministic** or **casual**.

If the signal repeats itself in a time interval proportional to a period T_0, it is called **periodic**. $f_0 = 1/T_0$ is the *fundamental frequency*. If $y(t)$ is a periodic signal of period T_0 and $x(t)$ is the expression for a single period, we can write that

$$y(t) = \sum_{n=-\infty}^{\infty} x(t - nT_0)$$

1.1 Energy and Power

The definitions will be given for continuous and discreet signals respectively.

Energy:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad E = \sum_{n=-\infty}^{\infty} |x_n|^2$$

Average power:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \qquad P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_n|^2$$

Average power on the interval T:

$$P_T = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \qquad P = \frac{1}{2N+1} \sum_{n=-N}^{N} |x_n|^2$$

Average power of a periodic signal:

$$P = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} |x(t)|^2 dt \qquad P = \frac{1}{N_o} \sum_{n=0}^{N_o-1} |x_n|^2$$

The rectangle

$$x(t) = rect(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| \geq \frac{1}{2} \end{cases}$$

See that $rect(t) = \theta(t + 1/2) - \theta(t - 1/2)$. Multiplying a signal $x(t)$ and the rectangle, we make some kind of filter.

1.2 Elementary operations on the signals

Delay: the signal $x(t-\tau)$ is delayed by τ with respect to $x(t)$. It is translated rigidly to the **right**.

Advance: the signal $x(t+\tau)$ is advanced by τ with respect to $x(t)$. It is translated rigidly to the **left**.

Scaling: the signal $x(at)$ is scaled by a with respect to $x(t)$, and it is **dilated** or **compressed** depending on $|a| < 1$ or $|a| > 1$.

THE STEP (*scalino*)

$$x(t) = u(t) = \theta(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \qquad u_n = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

1.3 The impulse

Dirac's delta can be defined this way:

$$\delta(t) = \lim_{T \to 0} \frac{1}{T} rect\left(\frac{1}{T}\right) = \lim_{T \to 0} \frac{1}{T}\left[u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)\right]$$

It has an infinitesimal width, infinite height and unitary area. It is the derivative of the step function.

$$A = \int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad \delta(t) = \frac{du(t)}{dt} \qquad u(t) = \int_{-\infty}^{t} \delta(t)dt$$

1.3.1 Properties

$$x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$$
$$x(t) \cdot \delta(t-\tau) = x(\tau) \cdot \delta(t-\tau)$$
$$\int_{-\infty}^{\infty} x(t)\delta(t-\tau)dt = x(\tau)$$
$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}$$
$$\delta(at) = \frac{1}{|a|}\delta(t), \quad \delta(-t) = \delta(t)$$

1.3.2 The discrete impulse

$$\delta_n = u_n - u_{n-1} = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

We can write the step function in this way[1]

$$u_n = \sum_{m=-\infty}^{n} \delta_m = \sum_{k=0}^{\infty} \delta_{n-k}$$

Some properties:

$$x_n \cdot \delta_n = x_0 \cdot \delta_n \qquad x_n \cdot \delta_{n-m} = x_m \cdot \delta_{n-m}$$

$$\boxed{\sum_{n=-\infty}^{\infty} x_n \cdot \delta_{n-m} = x_m}$$

1.4 Complex exponential

$$x(t) = \exp\{j(2\pi f_0 t + \varphi)\}$$
$$x_n = x(nT) = \exp\{j(2\pi f_0 nT + \varphi)\}$$

$\phi_0 = f_0 T$ is called the **normalized frequency**. In the discrete case, the vector rotates on the complex plane in $2\pi\phi_0$ steps every Ts. When the angular step is $> \pi$ or $< \pi$, a **frequency alias** exists, when $\phi < -1/2$ or $\phi > 1/2$. Consequently, $f = \phi/T$ is limited to the interval between $-\frac{1}{2T}$ and $\frac{1}{2T}$. The frequency $\frac{1}{2T}$ is called **Nyquist's frequency**. $1/T$ is the **sampling frequency**.

[1] $k = n - m$.

2 | Sistemi Lineari Tempo Invarianti

Linear: when the output generated from the linear combination of two or more inputs is equal to the linear combination of the outputs of each of those inputs.

$$O[x_1(t) + x_2(t)] = O[x_1(t)] + O[x_2(t)]$$

Time Invariant: when the output generated by a retarded signal is equal to the retarded output generated by the original signal.

$$O[x(t)] = y(t) \longrightarrow O[x(t-t_0)] = y(t-t_0)$$

We define the **response to the impulse** $h(t)$ as the output of the system when the input is an impulse.

$$h(t) = O[\delta(t)]$$

If the system is time-invariant, the form of the output is independent of the time on which the impulse has been applied.

$$h(t-\tau) = O[\delta(t-\tau)]$$

If the system is also linear:

$$O[a\delta(t) + b\delta(t-\tau_1) + c\delta(t-\tau_2)] = ah(t) + bh(t-\tau_1) + ch(t-\tau_2)$$

Any signal $x(t)$ can be represented as an integral sum of impulses:

$$\boxed{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)}$$

2.1 Convolutions

According to what he have discussed:

$$y(t) = O[x(t)] = O\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right] = \int_{-\infty}^{\infty} x(\tau)O[\delta(t-\tau)]d\tau =$$
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = x(t) * h(t)$$

The **integral of convolution** is defined as

$$\boxed{x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau}$$

2.1.1 Properties of convolutions

$$x(t) * h(t) = h(t) * x(t)$$
$$y(t - t_0) = x(t) * h(t - t_0) = x(t - t_0) * h(t)$$
$$g(t) * \delta(t) = \int_{-\infty}^{\infty} g(\tau)\delta(t - \tau)d\tau = g(t)$$
$$g(t) * \delta(t - t_0) = g(t - t_0)$$

2.1.2 Causality of the LTI systems

A LTI system is said to be **causal** if the output $y(t)$ for a instant $t = \bar{t}$ depends of the input $x(t)$ only for values $t \leq \bar{t}$. When the independent variable is time, a physical system must be causal. The condition that must be fulfilled to respect causality is:

$$\boxed{h(t) = 0 \text{ for } t < 0}$$

2.1.3 Effects of convolutions

Low-pass filter: the fast-varying components of the signal are eliminated through the convolution if $h(t)$ is slow-varying.

High-pass filter: the low-varying components of the signal are eliminated if $h(t)$ is fast-varying.

2.2 LTI discrete systems

In a discrete system, $y_n = O[x_n]$. The response to the impulse is: $h_n = O[\delta_n]$.

Any *discrete* signal can be written as a sum or impulses

$$x_n = \sum_{k=-\infty}^{\infty} x_k \delta_{n-k}$$

Therefore, as in the continuous case:

$$y_n = O[x_n] = O\left[\sum_{k=-\infty}^{\infty} x_k \delta_{n-k}\right] = \sum_{k=-\infty}^{\infty} x_k O[\delta_{n-k}] = \sum_{k=-\infty}^{\infty} x_k h_{n-k} = x_n * h_n$$

The **sum of convolution**:

$$\boxed{\sum_{k=-\infty}^{\infty} x_k h_{n-k} = x_n * h_n}$$

Properties:

$$
\begin{aligned}
x_n * y_n &= y_n * x_n && \text{Commutative}\\
x_n * [y_n * z_n] &= [x_n * y_n] * z_n && \text{Associative}\\
x_n * [y_n * z_n] &= x_n * y_n + x_n * z_n && \text{Distributive}\\
x_n * \delta_{n-m} &= x_{n-m}
\end{aligned}
$$

3 | Descrizione dei segnali e dei sistemi nelle frequenze

Both the time domain representation and the frequency domain have the same information. However, sometimes working in the frequency domain is much more simple. The **Fourier Analysis** allows us to study any signal as a combination of sinusoidals or **complex exponentials**.

3.1 Spectral characterization of signals

Indeed, any periodic signal $s(t)$ of period T can be decomposed in a discrete number of sinusoidals whose frequencies are multiples of $f = 1/T$. Those components are called **harmonics**. The representation of $s(t)$ in the domain of the frequencies of its harmonics constitutes the **signal's spectrum** $S(f)$.

3.1.1 Representation of periodic signals

The **periodic** signal $y(t)$ of period T_0 can be expressed as a **Fourier series**

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{j2\pi k f_0 t}$$

where $f_0 = 1/T_0$ and

$$Y_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} y(t)e^{-j2\pi k f_0 t}dt$$

If $y(t)$ is **real**, its Fourier series expansion has **hermitian symmetry**[1]

$$\boxed{Y_k = Y_{-k}^*}$$

As a consequence of this, the Fourier series of a real periodic signal can be written as a **sum of sines and cosines**.

$$y(t) = Y_0 + 2\sum_{k=1}^{\infty} \text{Re}\{Y_k\}\cos(2\pi k f_0 t) - \text{Im}\{Y_k\}\sin(2\pi k f_0 t)$$
$$y(t) = Y_0 + 2\sum_{k=1}^{\infty} |Y_k|\cos(2\pi k f_0 t + \theta_k)$$

[1]Complex conjugate symmetry.

3.1.2 Examples

Example 1: half-null square wave.

$$y(t) = \begin{cases} 0, & 0 \le t \le T_0/2 \\ 1, & T_0/2 < t \le T_0 \end{cases}$$

For this, $Y_0 = 1/2$ and

$$Y_k = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} e^{-j2\pi k f_0 t}dt = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} \cos(2\pi k f_0 t)dt = \frac{\sin\left(\pi\frac{k}{2}\right)}{\pi k}$$

We have only **odd components**, as $\sin\left(\pi\frac{k}{2}\right) = 0$ for $k = 2q$, $q \in \mathbb{Z}$.

Example 2: constant.

For a signal $y(t) = 1$, we have that

$$Y_k = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

Example 3: complex exponential

For $y(t) = e^{j2\pi f_0 t}$ we have that $Y_0 = 0$ and

$$Y_k = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases}$$

The Fourier analysis can be **extended** for all signals that change on time, periodic or not. The signal $s(t)$ can be represented on the frequency domain of its components, which constitute the **spectrum of the signal** $S(f)$.

A signal with a **wide band** varies fast on time, whereas a signal with a **tight band** varies slowly.

3.2 Frequency response of LTI systems

If the input signal of a LTI system is a complex exponential, the output will be a complex exponential with the **same frequency**, but different amplitude and phase. The **frequency response** $H(f)$ is a function of f that describes how the amplitude and phase of a complex exponential is modified when it goes through a LTI system.

If $x(t) = e^{j2\pi f_0 t}$, we have that

$$\begin{aligned} y(t) = x(t) * h(t) &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = e^{j2\pi f_0 t}\int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f_0 \tau}d\tau \\ &= x(t)\int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f_0 \tau}d\tau = e^{j2\pi f_0 t}H(f_0), \quad H(f_0) \in \mathbb{C} \end{aligned}$$

The frequency response can be obtained experimentally using k complex exponentials of frequency f_k.

$H(f)$ is a complex function of frequency which only depends on the system's response to the impulse $h(t)$.

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt$$

It introduces the concept of **bandwidth** (*banda passante*) of a LTI system. The module of $H(f)$ will have higher values in a frequency interval called bandwidth, and relatively smaller ones for the other frequencies. The complex exponentials with frequencies belonging to the BW will have the highest amplitudes in the output.

3.2.1 Fourier Transform

The operator that allows us to get $H(f)$ from $h(t)$ is called **Fourier Transform** ($\mathcal{F}$):

$$\boxed{X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt}$$

The inverse way can be done using the **Inverse Fourier Transform** ($\mathcal{F}^{-1}$):

$$\boxed{x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df}$$

It means that any signal $x(t)$ can be expressed as the sum (integral) of complex exponentials whose amplitudes and initial phases as a function of frequency are given by the Fourier Transform $X(f)$:

$$\text{Amplitude: } |X(f)|\, df \qquad \text{Initial phase: } \angle X(f)$$

Some considerations:

- $x(t)$ has an uniquely defined $X(f)$, and vice versa.
- For any value of f, $Y(f) = H(f)X(f)$.

3.2.2 Frequency response of an ideal low-pass filter

A system is called **low-pass filter** when the frequency response $H(f)$ has nonzero amplitude only in a symmetric band with respect to the origin.

An ideal low-pass filter with cutoff frequency f_c has as frequency response a rectangle of unitary amplitude and base $2f_c$: $H(f) = rect(f/2f_c)$.

3.2.3 Frequency response of an ideal high-pass filter

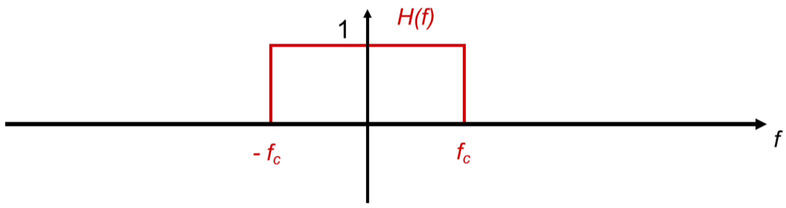
A system is said to be a **high-pass** filter when its response in frequency $H(f)$ has nonzero amplitude only for $f > f_c$ and $f < -f_c$.

An ideal high-pass filter with cutoff frequency f_c has as $H(f) = 1 - rect(f/2f_c)$.

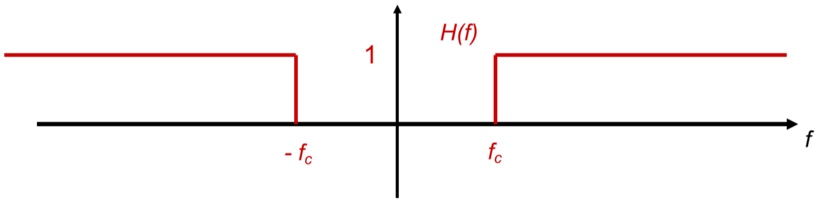
3.2.4 Frequency response of an ideal band-pass filter

When $H(f)$ has nonzero amplitude only in two bands symmetrically positioned with respect to the origin and centered in $-f_0$ and f_0, the system is called **band-pass filter**.

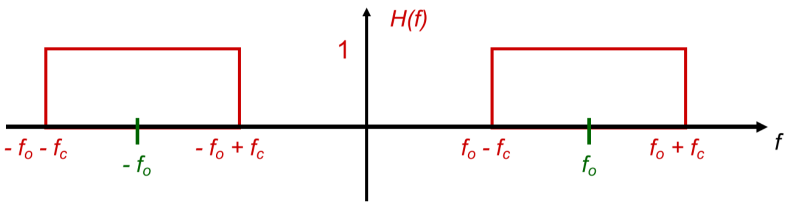
If it is ideal, with **central frequency** f_0 and **bandwidth** $2f_c$, its frequency response is two unitary-amplitude rectangles with base $2f_c$, centered in $-f_0$ and f_0.



(a) Low-pass.



(b) High-pass.



(c) Band-pass.

Figure 3.1: Frequency responses of ideal filters.

3.2.5 Discrete systems

If the input of a LTI system is a discrete complex exponential, the output will be a discrete complex exponential with the same frequency, but modified amplitude and phase.

Taking into account that $y_n = \sum_{k=-\infty}^{\infty} x_{n-k}h_k$, if $x_n = e^{j2\pi fnT}$ we have that

$$\begin{aligned} y_n &= \sum_{k=-\infty}^{\infty} e^{j2\pi f(n-k)T} h_k = e^{j2\pi fnT} \sum_{-\infty}^{\infty} e^{-j2\pi fkT} h_k \\ &= x_n \sum_{-\infty}^{\infty} e^{-j2\pi fkT} h_k = e^{j2\pi fnT} H(f) \end{aligned}$$

The operator that allows us to get $H(f)$ from h_n is called **Fourier Transform**, and it is the following one:

$$\boxed{X(f) = \sum_{n=-\infty}^{\infty} x_n e^{-j2\pi fnT}}$$

Note that the Fourier Transform of a discrete signal is periodic with period T.

$$X\left(f + \frac{k}{T}\right) = \sum_{-\infty}^{\infty} x_n e^{-j2\pi\left(f+\frac{k}{T}\right)nT} = \sum_{-\infty}^{\infty} x_n e^{-j2\pi fnT} e^{-j2\pi kn} = X(f)$$

The operator to do the opposite way is called **Inverse Fourier Transform**, and it is given by

$$\boxed{x_n = T \int_{-1/2T}^{1/2T} X(f) e^{j2\pi fnT} df}$$

Note that we only use *one period* of $X(f)$ to calculate the integral.

If we substitute the **normalized frequency** $\phi = f \cdot T$ we obtain these expressions for the preceding operations:

$$\begin{aligned} X(f) &= \sum_{n=-\infty}^{\infty} x_n e^{-j2\pi\phi n} \\ x_n &= \int_{-1/2}^{1/2} X(\phi) e^{j2\pi\phi n} d\phi \end{aligned}$$

The Fourier transform of a discrete signal in normalized frequency is periodic of **unit period**.

3.3 Properties of Fourier Transforms

1) **Linearity.**

$$\mathcal{F}\{ax(t) + by(t)\} = aX(f) + bY(f)$$

2) **Scaling.**

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|}X\left(\frac{f}{a}\right)$$

Particularly, when $a = -1$:

$$\mathcal{F}\{x(-t)\} = X(-f)$$

3) **Symmetry.** The TDF[2] of a **real signal** has **complex conjugate symmetry**.

$$X^*(f) = X(-f)$$

The real part and the module are even/symmetric, and the imaginary part and the phase odd/anti-symmetric.

$$\mathrm{Re}\{X(f)\} = \mathrm{Re}\{X(-f)\} \qquad |X(f)| = |X(-f)|$$
$$\mathrm{Im}\{X(-f)\} = -\mathrm{Im}\{X(f)\} \qquad \angle X(-f) = -\angle X(f)$$

Particular cases:

- $x(t) \in \mathbb{R}$ and even $\Rightarrow X(f) \in \mathbb{R}$ and even.
- $x(t) \in \mathbb{R}$ and odd $\Rightarrow X(f) \in \mathbb{I}$ and odd.

If the **signal** $x(t)$ is **imaginary**, there is also **symmetry**:

$$X^*(f) = -X(-f)$$

4) **Duality.** Given the signal $x(t)$ and its TDF $X(f)$, the following is fulfilled:

$$x(t) \xrightarrow{\mathcal{F}} X(f)$$
$$X(t) \xrightarrow{\mathcal{F}} x(-f)$$

5) **Value in the origin.** The TDF in $f = 0$ is equal to the integral of the signal over time. The value of the signal in $t = 0$ is equal to the integral of the TDF over the frequencies.

$$X(0) = \int_{-\infty}^{\infty} x(t)dt \qquad x(0) = \int_{-\infty}^{\infty} X(f)df$$

6) **Translation in time and frequency**.

$$\mathcal{F}\{x(t - t_0)\} = e^{-j2\pi f t_0}X(f) \qquad \mathcal{F}^{-1}\{X(f - f_0)\} = e^{j2\pi f_0 t}x(t)$$

[2] *Trasformata di Fourier.*

7) **Convolution:** Frequency multiplication.

$$\mathcal{F}\{x(t) * h(t)\} = \mathcal{F}\left\{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right\} = X(f)H(f)$$

It is an important property, as it allows to get the response $y(t) = x(t) * h(t)$ of a system working on the *frequency domain*: $y(t) = \mathcal{F}^{-1}\{X(f)H(f)\}$.

8) **Modulation:** Time multiplication.

$$\mathcal{F}\{x(t)y(t)\} = \int_{-\infty}^{\infty} X(\xi)Y(f-\xi)d\xi = X(f) * Y(f)$$

When a signal $x(t)$ is multiplied with another signal $y(t)$, it is said that one *modulates* the amplitude of the other. This allows to increase the frequency of the transmitted signal.
Check **quadrature amplitude modulation**. It allows to transmit two signals, modulating the amplitudes of two carrier waves, which are of the same frequency but out of phase with each other by 90° (*orthogonality/quadrature*). Thanks to that, at the receiver both waves can be coherently demodulated.

9) **Time derivative.**

$$\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j2\pi f\, X(f) \iff \mathcal{F}\{j2\pi t\, x(t)\} = -\frac{dX(f)}{df}$$

Be careful with the *constant values* that might be lost during derivation. Related with this, know that, related to the step function $u(t)$,

$$U(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) \neq \frac{1}{j2\pi f}$$

If the calculus is done without too much attention, the wrong expression is obtained. A good way to check results is that if the transform is symmetric, the original function must be so.

10) **Time integral.**

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$$

This can be shown using what we have said about $U(f)$ on the previous property. The dual of this property is:

$$\mathcal{F}\left\{\frac{-x(t)}{j2\pi t} + \frac{1}{2}x(0)\delta(t)\right\} = \int_{-\infty}^{f} X(\eta)d\eta$$

11) **Parseval's relation.** The energy of a signal is equal to the integral of the squared module of its TDF.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

So, we have two ways to compute the energy of a signal. $|X(f)|^2$ is called **spectral energy density**. This can be shown knowing that

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

which is eq. 3.14 from the book. It can be proved writing $x(t)$ as the inverse transform of $X(f)$. Remember that $\mathcal{F}\{x^*(t)\} = X^*(-f)$.

Autocorrelation function

Using Parseval's relation we can write that:

$$\int_{-\infty}^{\infty} X(f)X^*(f)e^{i2\pi f\tau}\, df = \int_{-\infty}^{\infty} x^*(t)x(t+\tau)\, dt$$

We define the *autocorrelation function* of the signal $x(t)$ as:

$$\boxed{R_x(\tau) = \int_{-\infty}^{\infty} x^*(t)x(t+\tau)\, dt}$$

It measures how similar a signal translated by τ is to itself. It is a particularly interesting measurement for the analysis of non-deterministic signals. We see that:

$$R_x(\tau) = \mathcal{F}^{-1}\{|X(f)|^2\} \qquad |X(f)|^2 = \mathcal{F}\{R_x(\tau)\}$$

It has the following properties:

1) The transform of autocorrelation is real, so it has hermitian symmetry:

$$R_x(\tau) = R_x^*(-\tau)$$

2) It can be written as a convolution:

$$R_x(\tau) = x(\tau) * x^*(-\tau)$$

3) $R_x(0) \geq |R_x(\tau)|$.
4) $R_x(0)$ is real and equal to the energy E of the signal $x(t)$ (see property 5).
5) If the signal $x(t)$ is **real**, $R_x(\tau)$ will be real and even symmetric.

Cross-correlation function

Using Parseval's relation once more, we have that

$$\int_{\infty}^{\infty} y(t+\tau)x^*(t)\, dt = \int_{-\infty}^{\infty} X(f)^*Y(f)e^{j2\pi f\tau}\, df$$

The first operator is called the *cross-correlation function* between signals $x(t)$ and $y(t)$. Normally we write it as

$$\boxed{R_{xy}(\tau) = \int_{-\infty}^{\infty} y(t+\tau)x^*(t)\,dt}$$

It can be written as

$$R_{xy}(\tau) = \mathcal{F}^{-1}\{X(f)^*Y(f)\}$$

So the transform of $R_{xy}(\tau)$ is $X^*(f)Y(f)$, the **cross-spectrum** of signals $x(t)$ and $y(t)$.

Properties:

1) $R_{xy}(\tau) = y(\tau) * x^*(-\tau)$.
2) $R_{xy}(\tau) = R^*_{yx}(-\tau)$.
3) If $x(t)$ and $y(t)$ are **real**, so is the cross-correlation, and $R_{xy}(\tau) = R_{yx}(-\tau)$.
4) If $y(t) = x(t - t_0)$ we have that

$$R_{xy}(\tau) = R_x(\tau - t_0)$$

An important application of the cross-correlation function is the measurement of the response to the impulse of a LTI system.

This in practice can not be directly done, as generating an impulse is not easy. If $y(t) = x(t) * h(t)$, we can write that

$$R_{xy}(\tau) = y(\tau) * x^*(-\tau) = h(\tau) * x(\tau) * x^*(-\tau) = h(t) * R_x(\tau) = h(\tau) * R_x(\tau)$$

We need to transmit a function $x(t)$ with a almost impulsive autocorrelation[3] $R_x(\tau) \simeq \delta(\tau)$. This ensures that

$$R_{xy}(\tau) \simeq h(\tau)$$

The energy we would need using this trick is much more small than the one needed to generate an impulse.

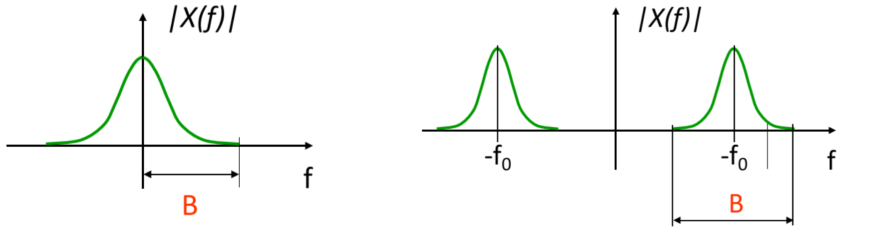
3.3.1 Band of a signal

The **band** B of a signal $x(t)$ is given by the frequency interval (measured on the positive semi-axis) on which $X(f)$ has non-zero values. For the cases when $X(f) \neq 0$, $\forall f$, the band corresponds to the frequency interval on which $X(f)$ is *significantly* different from 0.

[3] An example with this characteristic is the "chirp", defined as:

$$s(t) = \exp(j\pi k t^2)$$

Its frequency is a linear function of time: $f = kt$.



(a) **Low-pass $X(f)$ signal type, gathered around** $f = 0$.

(b) **Band-pass** type $X(f)$ signal, gathered around $f = \pm f_0$.

Figure 3.2: Following the definition of band, we consider two classes of signals.

3.3.2 Some important Fourier Transforms

The impulse

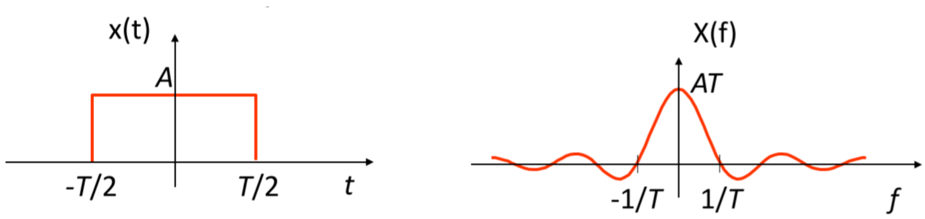
$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} \delta(t)e^{0}dt = \int_{-\infty}^{\infty} \delta(t)dt = 1$$

Thanks to property 4, we know that

$$\mathcal{F}\{1\} = \int_{-\infty}^{\infty} e^{-j2\pi ft}dt = \delta(-f) = \delta(f)$$

The rectangle

$$x(t) = A \cdot rect\left(\frac{t}{T}\right) \qquad \Longleftrightarrow \qquad X(f) = AT\frac{\sin(\pi fT)}{\pi fT}$$

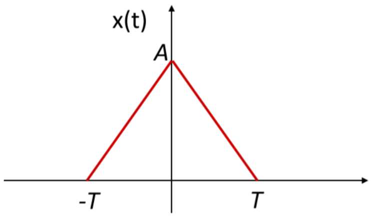


(a) $x(t)$

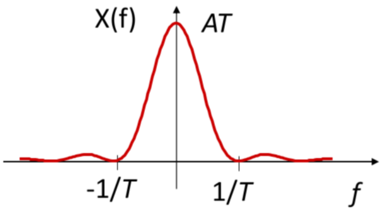
(b) $X(t)$

The triangle

$$x(t) = A \cdot tri_{2T}(t) = \begin{cases} A + \frac{A}{T}t, & -T \le t \le 0 \\ A - \frac{A}{T}t, & 0 < t \le T \end{cases} \qquad \Longleftrightarrow \qquad X(f) = AT^2\left(\frac{\sin(\pi fT)}{\pi fT}\right)^2$$



(c) $x(t)$

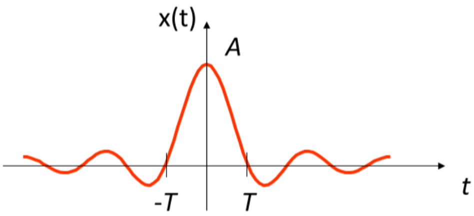


(d) $X(t)$

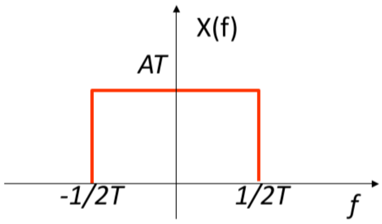
The cardinal sine

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$x(t) = A\frac{\sin(\pi/T\,t)}{\pi/T\,t} \qquad \Longleftrightarrow \qquad X(f) = AT \cdot rect(fT)$$



(e) $x(t)$



(f) $X(t)$

The complex exponential

$$X(f) = \mathcal{F}\{e^{j2\pi f_0 t}\} = \int_{-\infty}^{\infty} e^{j2\pi f_0 t} e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t} dt = \delta(f - f_0)$$

We can get the same conclusion applying the translation property:

$$\mathcal{F}\{1\} = \delta(f) \Rightarrow \mathcal{F}\{1 \cdot e^{i2\pi f_0 t}\} = \delta(f - f_0)$$

The cosine

$$\mathcal{F}\{\cos(2\pi f_0 t)\} = \mathcal{F}\left\{\frac{1}{2}e^{j2\pi f_0 t} + \frac{1}{2}e^{-j2\pi f_0 t}\right\} = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

The cosine is real and even, thus, its transform is also real and even.

The sine

$$\mathcal{F}\{\sin(2\pi f_0 t)\} = \mathcal{F}\left\{-\frac{j}{2}e^{j2\pi f_0 t} + \frac{j}{2}e^{-j2\pi f_0 t}\right\} = -\frac{j}{2}\delta(f - f_0) + \frac{j}{2}\delta(f + f_0)$$

The sine is real and odd, so its transform is purely imaginary and odd.

The Gaussian

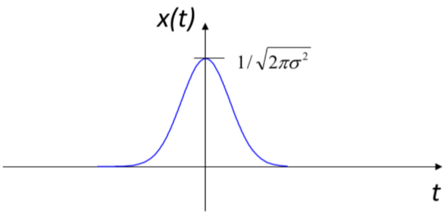
The Gaussian is the only function whose TDF is the function itself, $\mathcal{F}\{x(t)\} = X(f) = x(f)$:

$$x(t) = e^{-\pi t^2} \qquad \Longleftrightarrow \qquad X(f) = e^{-\pi f^2}$$

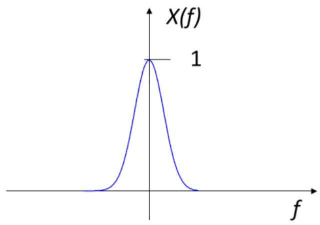
For the formula used in statistics, the normal distribution:

$$x(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} \qquad \Longleftrightarrow \qquad X(f) = e^{-2\pi^2\sigma^2 f^2}$$

This formula is gotten using the scaling property.



(g) $x(t)$



(h) $X(t)$

3.3.3 Impulse responses of ideal filters

We have seen the frequency responses of ideal filters in figures 3.1a, 3.1b and 3.1c. With the tools we have just gotten, we can compute the response to impulses of these systems.

Ideal low-pass

$$h(t) = \frac{\sin(2\pi f_c t)}{\pi t}$$

Ideal high-pass

$$h(t) = \delta(t) - \frac{\sin(2\pi f_c t)}{\pi t}$$

Ideal band-pass

$$h(t) = \frac{2\sin(2\pi f_c t)}{\pi t}\cos(2\pi f_0 t)$$

3.3.4 Properties of TDF of discrete signals

1) **Linearity**.
2) **Symmetry**. The TDF of a real signal has complex conjugate symmetry.

3) **Value in the origin.**

$$\tilde{X}(0) = \sum_{n=-\infty}^{\infty} x_n \qquad x_0 = T\int_{-1/2T}^{1/2T} \tilde{X}(f)df$$

4) **Translations.**

$$\mathcal{F}\{x_{n-m}\} = \tilde{X}(f)e^{-j2\pi fmT} \qquad \mathcal{F}\{x_n e^{+j2\pi f_0 nT}\} = \tilde{X}(f - f_0)$$

The tilde expresses explicitly the periodicity of the TDF of a discrete signal. See that if $f_0 = \frac{1}{T}$ (T being the period of the TDF), the translation has no effect.

5) **Convolution:** frequency multiplication. The TDF of the convolution of two discrete signals is equal to the product of the TDF of the two signals.

$$y_n = x_n * h_n = \sum_{k=-\infty}^{\infty} x_{n-k}h_k \longrightarrow Y(f) = \sum_{-\infty}^{\infty} h_k\tilde{X}(f)e^{-j2\pi fTk} = \tilde{X}(f)H(f)$$

6) **Modulation:** Time multiplication. The TDF of the product of two discrete signals is equal to the *circular/periodic convolution* of the two periodic TDFs multiplied by T.

$$\mathcal{F}\{x_n h_n\} = T\int_{-1/2T}^{1/2T} \tilde{X}(\xi)H(f-\xi)d\xi \longrightarrow Y(f) = \tilde{X}(f) \circledast H(f)$$

7) **Parserval's relation.** The energy of a signal is equal to the integral of the squared module of the TDF in one period, multiplied by T

$$\sum_{-\infty}^{\infty} |x_n|^2 = T\int_{-1/2T}^{1/2T} |\tilde{X}(f)|^2 df$$

In normalized frequency:

$$\sum_{-\infty}^{\infty} |x_n|^2 = \int_{-1/2}^{1/2} |\tilde{X}(\phi)|^2 d\phi$$

To arrive to this formulas, we need the relation

$$\sum_{n=-\infty}^{\infty} x_n y_n = T\int_{-1/2T}^{1/2T} X(-f)Y(f)\, df$$

Indeed, if $z_n = x_n y_n$, the sum of the samples z_n is equal to $Z(0)$. Besides,

$$Z(f) = T\int_{-1/2T}^{1/2T} X(f-\eta)Y(\eta)d\eta$$

For $f = 0$ we obtain the previous formula. Knowing that $\mathcal{F}\{y_n^*\} = Y^*(-f)$, we have that

$$\sum_{n=-\infty}^{\infty} x_n y_n = T\int_{-1/2T}^{1/2T} X(f)Y^*(f)\, df$$

Autocorrelation sequence and spectral energy density

From Parseval's relation:

$$T \int_{-1/2T}^{1/2T} X(f)X^*(f)e^{i2\pi fTk}\, df = \sum_{n=-\infty}^{\infty} x_n^* x_{n+k} = R_x[k]$$

We deduce that the transform of the *correlation sequence* $R_x[k]$ is equal to $|X(f)|^2$:

$$\mathcal{F}\{R_x[k]\} = |X(f)|^2$$

All the properties we saw for the continuous case are still valid.

Cross-correlation function

We can write that

$$T \int_{-1/2T}^{1/2T} X(f)Y^*(f)e^{i2\pi fTk}\, df = \sum_{n=-\infty}^{\infty} y_n^* x_{n+k} = R_{xy}[k]$$

The second summation takes the name of *cross-correlation sequence* between x_n and y_n. Its transform gives the *cross-spectrum* of the sequence:

$$\mathcal{F}\{R_{xy}[k]\} = X(f)Y^*(f)$$

The properties are the same we had in the continuous case.

Tempo	**Frequenza**
$x(t)$	$X(f)$
$y(t)$	$Y(f)$
$ax(t) + by(t)$	$aX(f) + bY(f)$
$x(0)$	$\int_{-\infty}^{\infty} X(f)df$
$\int_{-\infty}^{\infty} x(t)dt$	$X(0)$
$x(at)$ con a reale	$\frac{1}{\|a\|}X(\frac{f}{a})$
$x(t-t_0)$	$X(f)e^{-j2\pi f t_0}$
$x(t)e^{j2\pi f_0 t}$	$X(f-f_0)$
$x(t) * y(t)$	$X(f)Y(f)$
$x(t)y(t)$	$X(f) * Y(f)$
$\frac{dx(t)}{dt}$	$j2\pi f X(f)$
$-j2\pi t x(t)$	$\frac{\mathrm{d}X(f)}{\mathrm{d}f}$
$\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau$	$\frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$
$x^*(t)$	$X^*(-f)$
$\int_{-\infty}^{\infty} \|x(t)\|^2\mathrm{d}t$	$\int_{-\infty}^{\infty} \|X(f)\|^2 df$
$\int_{-\infty}^{\infty} x^*(t)x(t+\tau)dt = R_x(\tau)$	$\|X(f)\|^2$
If $x(t)$ is real	$X(-f) = X^*(f)$ $\mathrm{Re}\{X(f)\} = \mathrm{Re}\{X(-f)\}$ $\mathrm{Im}\{X(f)\} = -\mathrm{Im}\{X(-f)\}$ $\|X(f)\| = \|X(-f)\|$ $\angle X(f) = -\angle X(-f)$

Tempo	**Frequenza**	**Frequenza normalizzata**
$x_n = x(nT)$	$X(f)$ periodica $\frac{1}{T}$	$X(\phi)$ periodica 1
$y_n = y(nT)$	$Y(f)$ periodica $\frac{1}{T}$	$Y(\phi)$ periodica 1
$ax_n + by_n$	$aX(f) + bY(f)$	$aX(\phi) + bY(\phi)$
$x(0)$	$T\int_{-1/(2T)}^{1/(2T)} X(f)$	$\int_{-1/2}^{1/2} X(\phi)d\phi$
$\sum_{n=-\infty}^{\infty} x_n$	$X(0)$	$X(0)$
$x_{n-n_\circ}$	$X(f)e^{-j2\pi fTn_\circ}$	$X(\phi)e^{-j2\pi\phi n_o}$
$x_n e^{j2\pi f_0 Tn} = x_n e^{j2\pi\phi_0 n}$	$X(f - f_0)$	$X(\phi - \phi_0)$
$x_n * y_n$	$X(f)Y(f)$	$X(\phi)Y(\phi)$
$x_n y_n$	$\int_{-1/2T}^{1/2T} X(\vartheta)Y(f-\vartheta)d\vartheta$	$\int_{-1/2}^{1/2} X(\vartheta)Y(\phi-\vartheta)d\vartheta$
x_n^*	$X^*(-f)$	$X^*(-\phi)$
$\sum_{n=-\infty}^{\infty} \|x_n\|^2$	$T\int_{-1/2T}^{1/2T} \|X(f)\|^2 df$	$\int_{-1/2}^{1/2} \|X(\phi)\|^2 d\phi$
$\sum_{n=-\infty}^{\infty} x_n^* x_{n+k} = R_x[k]$	$\|X(f)\|^2$	$\|X(\phi)\|^2$
If $x(t)$ is real	$X(-f) = X^*(f)$ $\text{Re}\{X(f)\} = \text{Re}\{X(-f)\}$ $\text{Im}\{X(f)\} = -\text{Im}\{X(-f)\}$ $\|X(f)\| = \|X(-f)\|$ $\angle X(f) = -\angle X(-f)$	$X(-\phi) = X^*(\phi)$ $\text{Re}\{X(\phi)\} = \text{Re}\{X(-\phi)\}$ $\text{Im}\{X(\phi)\} = -\text{Im}\{X(-\phi)\}$ $\|X(\phi)\| = \|X(-\phi)\|$ $\angle X(\phi) = -\angle X(-\phi)$

List of useful continuous transforms

Tempo $x(t)$	**Frequenza** $X(f) = \mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$\delta(f)$
$rect(t/T)$	$\dfrac{\sin(\pi T f)}{\pi f}$
$\dfrac{\sin(\pi B t)}{\pi t}$	$rect(f/B)$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$e^{-at} \cdot u(t)$	$\dfrac{1}{a + j2\pi f}$
$\dfrac{1}{a + j2\pi t}$	$e^{af} \cdot u(-f)$
$t \cdot e^{-at} \cdot u(t)$	$\dfrac{1}{(a + j2\pi f)^2}$
$e^{j2\pi f_0 t}$	$\delta(f - f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
$\sin(2\pi f_0 t)$	$\frac{-j}{2}\delta(f - f_0) + \frac{j}{2}\delta(f + f_0)$
$\cos(2\pi f_0 t + \theta)$	$\frac{1}{2}e^{j\theta}\delta(f - f_0) + \frac{1}{2}e^{-j\theta}\delta(f + f_0)$
$\left(\dfrac{\sin(\pi t)}{\pi t}\right)^2$	$tri(f) = rect(f) * rect(f)$

List of useful discrete transforms

Remember that duality theorem does not exist. $u_n - u_{n-N}$ is a rectangle between $n = 0$ and $n = N-1$. If N is odd, $N-1$ is even and $k_0 = \frac{N-1}{2}$ is integer: $e^{-j\pi\phi(N-1)} = e^{-j2\pi\phi k_0}$ is an usual time translation. Instead, if N is even, we do not get a typical translation for the transform of the rectangle. This has to do with the fact that we can only center a rectangle in the origin if N is odd.

Tempo x_n	**Frequenza** $X(\phi) = \mathcal{F}\{x_n\}$
δ_n	$1\ (T = 1)$
$a^n \cdot u_n$	$\frac{1}{1-ae^{-j2\pi\phi}}\ (T = 1)$
$u_n - u_{n-N}$	$e^{-j\pi\phi(N-1)} \cdot \frac{\sin(\pi N\phi)}{\sin(\pi\phi)} (T = 1)$
1	$\sum_{k=-\infty}^{\infty} \frac{1}{T}\delta\left(f - \frac{k}{T}\right) = \sum_{n=-\infty}^{\infty} 1 \cdot e^{j2\pi fnT}$
	$\sum_{k=-\infty}^{\infty} \frac{1}{T}\delta\left(\frac{\phi}{T} - \frac{k}{T}\right) = \sum_{k=-\infty}^{\infty} \frac{1}{T}T\delta(\phi - k) = \sum_{k=-\infty}^{\infty} \delta(\phi - k)$

Simplest **low-pass** filter we can build for discrete signals:

$$h_n = \frac{1}{2}\delta_n + \frac{1}{2}\delta_{n-1}$$

Simplest **high-pass** filter:

$$h_n = \frac{1}{2}\delta_n - \frac{1}{2}\delta_{n-1}$$

This two examples are better analyzed in the notes you took on class.

4 | Del tempo continuo a quello discreto

4.1 Digitization of signals

In the current storage and transmission systems, the input signals are of numeric type, normally represented in binary format. However, the majority of the signals from reality are continuous both in time (*sampling*) and amplitude (quantization). To represent there as numeric signals, it is necessary to **discretize** them both in time and amplitude.

4.2 Sampling in time

The *sampling step/period* is represented by T, and $f_c = 1/T$ is called the *sampling frequency*. If we sample the signal $x(t)$, we obtain the sequence of samples $x(nT)$.

See that the same sequence is obtained for any of the following signals, remembering that $e^{j2\pi n} = 1,\ \forall\, n$:

$$x_k(t) = x(t)e^{j2\pi k f_c t} \quad \Longrightarrow \quad x_k(nT) = x(nT)e^{j2\pi k f_c nT} = x(nT)$$

In general, it is not possible to say which of the signals $x_k(t)$ (or their linear combinations) have generated the samples $x(nT)$.

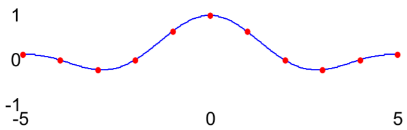
Consider the following example ($T = 1,\ f_c = 1$):

$$(a) \quad x(t) = \frac{\sin(0.5\pi t)}{0.5\pi t}$$

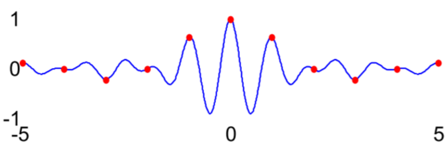
Two possible linear combinations are:

$$(b) \quad \frac{x_1(t) + x_{-1}(t)}{2} = x(t)\cos(2\pi t) \qquad (c) \quad \frac{x_2(t) + x_{-2}(t)}{2} = x(t)\cos(4\pi t)$$

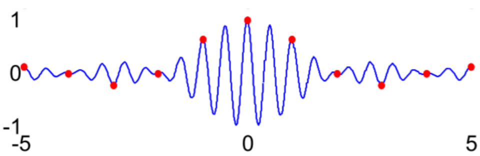
The sequence of samples we would obtain is the same in the three cases:



(a)



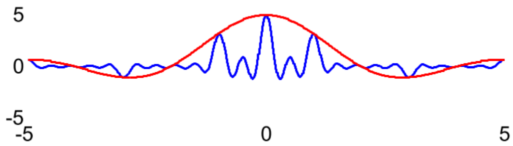
(b)



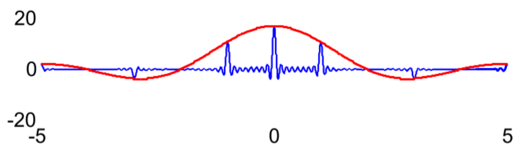
(c)

Continuing with the same $x(t)$, let's see what happens when summing it with the signals $x_k(t) = x(t)e^{j2\pi kt}$:

(d) $x_0(t)$



(e) $\sum_{k=-2}^{2} x_k(t)$



(f) $\sum_{k=-8}^{8} x_k(t)$

As the number elements in the sum K increases, the result of the sum gets closer to $K \cdot c(nT)$ for $t = nT$ and tends to zero for the rest of the points. Each element of the sum corresponds to a different replica of $X(f)$ centered in the frequency kf_c. Therefore the sum of infinite replicas in frequency $X(f)$ generates an impulsive signal proportional to

$$\sum_n x(nT)\delta(t - nT)$$

Therefore the spectrum of a signal sampled using impulses with time-step T is **periodic** with step $f_c = 1/T$ in frequency.

We define the **sampled signal** (*segnale campionato*) $x_c(t)$ as:

$$x_c(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

This operation is called *ideal sampling*, and $x_c(t)$ a *signal sampled ideally*. $x_c(t)$ is still a continuous signal in time, and our goal will be to show that is possible to reconstruct $x(t)$ when T is sufficiently small.

Using the modulation property of Fourier Transform, we see that:

$$X_c(f) = X(f) * \sum_{n=-\infty}^{\infty} e^{-j2\pi nTf} = X(f) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$$

The last step has been done using the formula (3.41) from the book. Finally, we get that:

$$X_c(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(f - \frac{k}{T}\right) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j2\pi fnT}$$

Showing that $X_c(f)$ is periodic with period $1/T$. $X_c(f) = \tilde{X}(f) = \mathcal{F}\{x_n\}$.

4.3 Sampling theorem

If the band B of $X(f)$ is limited between $f = -\frac{f_s}{2} = -\frac{1}{2T}$ and $f = \frac{f_s}{2} = \frac{1}{2T}$, this is, if $f_s = \frac{1}{T} > B$ or $T < \frac{1}{2f_{max}}$:

- The replicas of $X(f)$ do not overlap (there is no frequency ambiguity/alias).
- The Fourier Transform of the signal $x(t)$ can be obtained from the sampled signal, multiplying $X_c(f)$ with a rectangular function $H(f)$ of amplitude T between $f = -\frac{1}{2T}$ and $f = \frac{1}{2T}$.

The condition can be also formulated as:

$$f_{max} < \frac{1}{2T} = \frac{f_s}{2} = f_{Ny}$$

where f_{Ny} is called *Nyquist frequency*.

Teorema del campionamento: *un segnale reale tempo-continuo $x(t)$ può essere ricostruito esattamente dai suoi campioni $x_c(t)$ se la frequenza di campionamento f_s è maggiore del doppio della frequenza massima di $x(t)$.*

Multiplying in frequency with a rectangle of base f_c and amplitude T is equivalent to convolving in time with a cardinal sine that has unitary amplitude in $t = 0$ and null in $t = nT$. $H_R(f)$ is a low-pass filter called **reconstruction filter**:

$$H_R(f) = \frac{1}{f_s} rect(\frac{f}{f_s}) = T\, rect(fT)$$

$$x(t) = x_c(t) * \frac{\sin\left(\pi \frac{t}{T}\right)}{\pi \frac{t}{T}}$$

In reality, a perfect cardinal sine (infinite duration) can not be produced, and the used reconstruction filters have softer transitions. It is enough to add cardinal sines centered in $t = nT$, with $A_{max} = x(nT)$ and zeroes in $t = mT$ for all $m \neq n$. Consequently, it is not safe to try to reconstruct the signal $x(t)$ with the minimum f_s that the theorem gives: in practice, the sampling frequency we used is around 10% bigger than the one given by the sampling theorem[1].

[1] In CDs, $f_s = 44.1kHz$ for $f_{max} = 20kHz$, even if the theorem requires $f_s < 40kHz$.

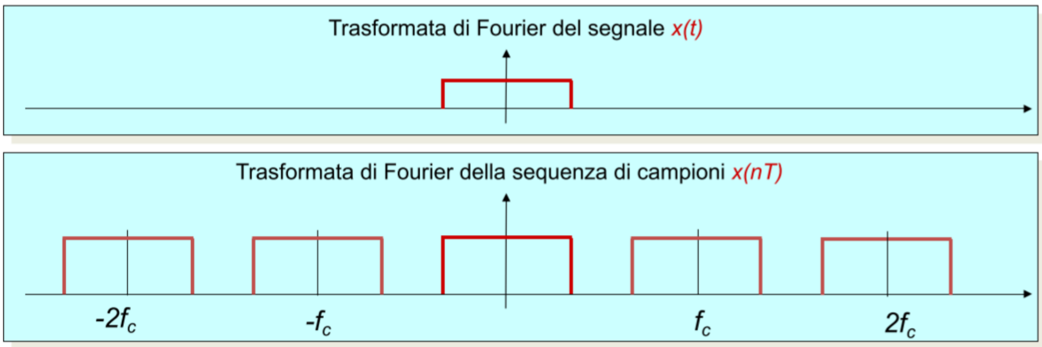


Figure 4.1: The Fourier Transform of a signal constituted of a sequence of samples $x(nT)$ is defined in the same way of the transform of the signal $\sum_n x(nT)\delta(t - nT)$. Therefore, the Fourier transform of the sequence $x(nT)$ is equal (multiplied by f_c) to the transform of the time-continuous signal $x(t)$ replicated in frequency infinite times, with step f_c.

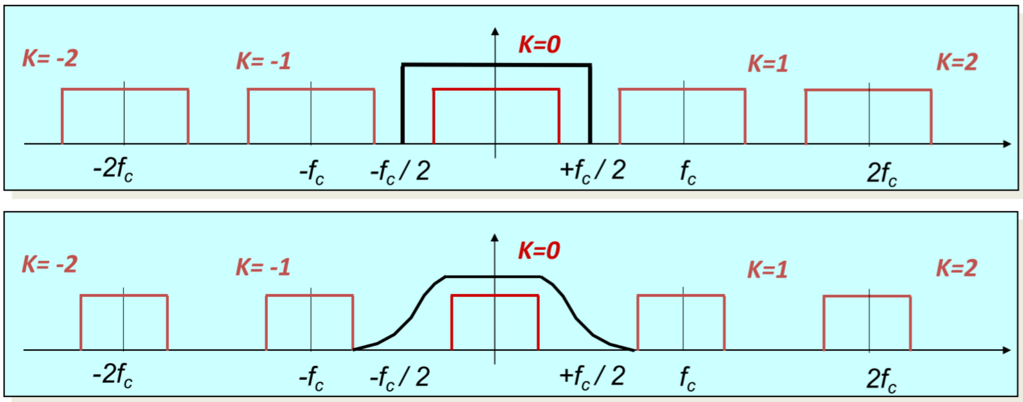


Figure 4.2: To obtain the Fourier transform of the time-continuous signal from the transform of the signal formed by samples, it is necessary to delete all the spectral replicas and leave only the one for $k = 0$. This can be done multiplying the transform of $x(nT)$ by a rectangle of amplitude T and duration between $f = \pm\frac{f_c}{2}$; alternatively, compute the convolution of its response with a cardinal sine-like signal. If the Nyquist frequency $f_c/2$ is bigger than the maximum frequency of the signal, the low-pass filter can have softer transitions.

4.4 Practical systems for reconstruction

In theory, to get $x(t)$ from the numeric sequence x_n it is needed to first create a sequence of impulses $x_c(t) = \sum_{n=-\infty}^{\infty} x_n\delta(t - nT)$ and to use a low-pass filter on it. As we have already said, in practice we can not create ideal impulses, so other kinds of signals will be needed.

4.4.1 Reconstruction with maintenance

An alternative to the impulse $\delta(t)$ is a rectangle $rect(t/T)$. The sequence x_n becomes a sequence of rectangular impulses:

$$x_h(t) = \sum_{n=-\infty}^{\infty} x_n \, rect\left(\frac{t-nT}{T}\right)$$

This technique is said to be *with maintenance* (*sample and hold*) as the amplitude of the sample x_n is maintained. It can be shown (page 98 in the book) that

$$x_h(t) = x_c(t) * rect\left(\frac{t}{T}\right)$$

The Fourier transform will therefore be:

$$X_h(f) = X_c(f)\frac{\sin(\pi f T)}{\pi f} = X_c(f)\frac{\sin\left(\pi \frac{f}{f_s}\right)}{\pi f}$$

This function shows the same replicas as $X_c(f)$, but modulated with a cardinal sine whose value is T in $f = 0$ and is null for $f = k/T = kf_s$. The reconstruction filter must be designed in a way that eliminates the replicas, but also cancels the effect of the cardinal sign. The response of the filter takes the form:

$$H_R(f) = \frac{\pi f}{\sin\left(\pi \frac{f}{f_s}\right)} rect\left(\frac{f}{f_s}\right)$$

4.4.2 Reconstruction with oversampling

Examining the previous formula for $H_R(f)$ we see that for $f \ll f_s$ the effect of the modulation introduced by the cardinal sine is negligible. If the band of the signal is $B \ll f_s/2$, the response of the reconstruction filter must be practically constant in the band B, and must go to zero in $f_s - B$, where the first replica starts.

Sampling the signal $x(t)$ of band B with a sampling frequency $f_s \gg 2B$ generates a increase of the amount of samples, but consequently, of the cost of memorization and transmission.

In practice, the lowest possible frequency is used for sampling, and the highest for the reconstruction phase, **interpolating** numerically the samples x_n before reconstructing $x(t)$. To understand the process, remember that the reconstructed signal $x_R(t)$ is obtained as:

$$x_R(t) = x_c(t) * \frac{\sin\left(\pi \frac{t}{T}\right)}{\pi \frac{t}{T}}$$

Using the expressions we have previously obtained, the signal reconstructed ideally:

$$x_R(t) = \left[\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) \right] * \frac{\sin\left(\pi \frac{t}{T}\right)}{\pi \frac{t}{T}} = \sum_{n=-\infty}^{\infty} x_n \cdot \frac{\sin\left(\pi \frac{t-nT}{T}\right)}{\pi\left(\frac{t-nT}{T}\right)}$$

This allows us to write, for example, the formula for $x(m\frac{T}{8})$ as a function of the known samples x_n. Substituting $t = m\frac{T}{8}$ in the previous formula:

$$x\left(m\frac{T}{8}\right) = \sum_{n=-\infty}^{\infty} x_n \cdot \frac{\sin\left(\pi \frac{mT/8-nT}{T}\right)}{\pi\left(\frac{mT/8-nT}{T}\right)} = \sum_{n=-\infty}^{\infty} x_n \cdot \frac{\sin\left(\pi(\frac{m}{8}-n)\right)}{\pi(\frac{m}{8}-n)}$$

This operation is called *8:1 oversampling* of the sequence x_n, and it can be generalized to any M:1 this way:

$$x\left(m\frac{T}{M}\right) = \sum_{n=-\infty}^{\infty} x_n \cdot \frac{\sin\left(\pi(\frac{m}{M}-n)\right)}{\pi(\frac{m}{M}-n)}$$

5 | The Discrete Fourier Transform

Note: we will use $\tilde{X}(f)$ to indicate the Fourier transform of x_n, which can not be confused with $X(f)$, the Fourier transform of the time continuous signal $x(t)$.

5.1 Sampling in time and frequency

In the examples from previous chapters we have seen that a time continuous signal of finite duration has a Fourier transform with infinite bandwidth. However, usually almost all the spectral energy density confined to a limited bandwidth. Dually, signals with limited band have always infinite duration, but the majority of their energy is usually limited to a finite interval. So, in practice, we can admit the existence of signals with finite duration and band, at least as a first approximation.

Consider the signal $x(t)$ of limited duration T_0 ($0 \leq t < T_0$), whose Fourier transform $X(f)$ has limited band B ($-B < f < B$). If we sample $x(t)$ with a sampling interval of $T < \frac{1}{2B}$, the sequence $x_n = x(nT)$ is obtained. Its Fourier transform has the following formula:

$$\tilde{X}(f) = \sum_{n=-\infty}^{\infty} x_n e^{-j2\pi fnT} = \frac{1}{T} \sum_{m=-\infty}^{\infty} X\left(f - \frac{m}{T}\right)$$

As a consequence of the limited duration T_0 of the signal $x(t)$, the sequence x_n will have only $N = \frac{T_0}{T}$ non-zero samples (from $n = 0$ to $n = N-1$). So its Fourier transform can be written as:

$$\tilde{X}(f) = \sum_{n=0}^{N-1} x_n e^{-j2\pi fnT}$$

If we now sample in frequency $\tilde{X}(f)$ with a sampling interval[1] $\frac{1}{T_0} = \frac{1}{NT}$, we obtain a sequence in frequency related to x_n by the following relation:

$$\tilde{X}_k = \tilde{X}(f)\Big|_{f=\frac{k}{NT}} = \sum_{n=0}^{N-1} x_n e^{-j2\pi \frac{kn}{N}}$$

The sequence $\tilde{X}_k$ is periodic every N samples (given that $\tilde{X}(f)$ is periodic of period $\frac{1}{T}$), and therefore it is completely defined by a sequence X_k of N samples included, for example, between $k = 0$ and $k = N-1$:

Discrete Fourier Transform (DFT)

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi \frac{kn}{N}} \quad \text{computed for} \quad 0 \leq k \leq N-1$$

[1] $1/T_0$ is the longest sampling interval (in frequency) that allows to avoid temporal alias.

It can be shown (book, page 106) that there exists a analogous expression to the DFT that allows to find the N samples of x_n using the N samples of X_k. This relation is called *Inverse Discrete Fourier Transform*, and has the following formula:

Inverse Discrete Fourier Transform (DFT)

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi\frac{kn}{N}} \quad \text{computed for} \quad 0 \le n \le N-1$$

This can be extended to values of n that go from $-\infty$ to ∞, and this way a periodic signal $\tilde{x}_n$ of period N is obtained, whose base is the signal x_n:

$$\tilde{x}_n = \sum_{p=-\infty}^{\infty} x_{n-pN}$$

5.2 Utility of the DFT

Using the DFT, we have seen that a sequence x_n with a finite number of samples in the time domain is associated to a sequence X_k with the same number of samples in the domain of the frequency. The practical significance of this fact is huge, as it allows to elaborate numeric signals both in time and frequency domains.

In fact, the sequence X_k (DFT of the sequence x_n), contains **all the information**[2] of $X(f)$ (Fourier transform of the same sequence x_n, continuous in frequency).

It is important to remark that both X_k and x_n represent a period of N samples (conventionally between 0 and $N-1$) of **periodic** sequences.

5.3 Format of the DFT

It is immediate to obtain the existing relation between the sequence X_k, DFT of the sequence x_n, and $X(f)$ and $X(\phi)$, transform and normalized transform of the same frequency x_n:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi\frac{nk}{N}} = X(f)|_{f=\frac{k}{NT}} = X(\phi)|_{\phi=\frac{k}{N}}$$

One consequence of this choice is that with even N, the maximum normalized frequency $\phi = 1/2$ (Nyquist frequency) is obtained with the sample $k = N/2$ of the

[2]Remember that $X(f)$ has been sampled fulfilling the sampling theorem, and consequently, it is completely rebuildable starting from the samples.

DFT. Please note that normally the continuous transform of a sequence is represented between $\phi = -1/2$ and $\phi = 1/2$, while with the DFT format the period is represented with samples that go form $\phi = 0$ to $\phi = 1 - 1/N$. When N is odd, Nyquist frequency does not correspond to a sample of X_k.

5.4 Circular representation

We have already said that x_n has to be seen as a period of the periodic sequence $\tilde{x}_n$ and X_k as a period of the periodic sequence $\tilde{X}_k$. Because of this, it is useful to represent the sequences x_n and X_k with their samples forming a circumference, so that after the last sample ($n = N - 1$) we find again the first ($n = 0$).

This idea allows us to work when the samples do not go from $n = 0$ to $n = N - 1$, reordering them. This operation is called *circular delay.* This can be done in MatLab using the functions `FFT` and `SHIFT`.

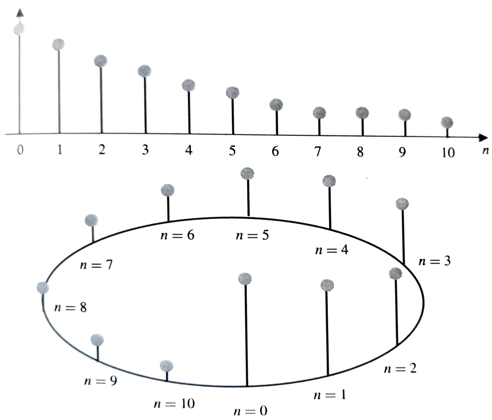


Figure 5.1: Circular representation.

5.5 Some examples

1) $x_n = 1$

$$X_k = \sum_{n=0}^{N-1} 1 \cdot e^{-j2\pi \frac{nk}{N}} = N\delta_k$$

2) $x_n = e^{j2\pi \frac{k_0}{N} n}$

$$X_k = \sum_{n=0}^{N-1} e^{j2\pi \frac{k_0}{N} n} \cdot e^{-j2\pi \frac{nk}{N}} = \sum_{n=0}^{N-1} e^{-j2\pi \frac{n}{N}(k-k_0)} = N\delta_{k-k_0}$$

3) $\cos(2\pi \frac{k_0}{N} n) = \frac{1}{2} e^{j2\pi \frac{k_0}{N} n} - \frac{1}{2} e^{-j2\pi \frac{k_0}{N} n}$

$$X_k = \frac{N}{2}\delta_{k-k_0} + \frac{N}{2}\delta_{k+k_0}$$

The sub-index $k + k_0$ is not appropriate for a computer, it can not check negative frequencies. Using the circular shift, $k + k_0 = k - (-k_0) = k - (N - k_0)$

$$X_k = \frac{N}{2}\delta_{k-k_0} + \frac{N}{2}\delta_{k-(N-k_0)}$$

5.6 Properties of the DFT

5.6.1 Linearity

$$DFT\{ax_n + by_n\} = aX_k + bY_k$$

5.6.2 Symmetry

Same that in the Fourier transform's case, also the DFT $\tilde{X}_k$ of a **real** signal x_n has *complex conjugate symmetry*:

$$\tilde{X}_{-k} = \tilde{X}_k^*$$

For the DFT X_k defined in $0 \le k \le N - 1$, we have that

$$X_{N-k} = X_k^* \quad \Longrightarrow \quad \begin{cases} \mathrm{Re}\{X_{N-k}\} = \mathrm{Re}\{X_k\} \\ \mathrm{Im}\{X_{N-k}\} = -\mathrm{Im}\{X_k\} \end{cases}$$

with $1 \le k \le \frac{N}{2} - 1$ for even N, and $1 \le k \le \frac{N-1}{2}$ for odd N.

5.6.3 Initial values

Using the formulas of DFT and IDFT, it is immediate to show that

$$X_0 = \sum_{n=0}^{N-1} x_n \qquad x_0 = \frac{1}{N}\sum_{k=0}^{N-1} X_k$$

5.6.4 Circular translation

The change of variables $\nu = n - n_0$ can be used to see that:

$$\mathrm{DFT}(x_{n-n_0}) = \sum_{n=0}^{N-1} x_{n-n_0} e^{-j2\pi \frac{nk}{N}} = \sum_{n=0}^{N-1} x_n e^{-j2\pi \frac{(n+n_0)k}{N}} = X_k\, e^{-j2\pi \frac{n_0}{N} k}$$

If $n_0 = N$, we get $x_{n-N} = x_n$.

Analogously, it can be shown that:

$$\text{IDFT}(X_{k-k_0}) = x_n\, e^{j2\pi \frac{k_0}{N} n}$$

For $k_0 = N/2$, we have $e^{j2\pi n} = (-1)^n$.

5.6.5 Circular convolution

Let x_n and y_n be two sequences with N samples with DFT X_k and Y_k. The page 111 of the book shows that:

$$\text{DFT}(x_n \circledast y_n) = X_k \cdot Y_k$$

Where the *circular convolution* $x_n \circledast y_n$ is given by:

$$x_n \circledast y_n = \sum_{m=0}^{N-1} x_m \tilde{y}_{n-m}$$

If $z_n = x_n * y_n$ is the linear convolution, the circular convolution can be written as:

$$\tilde{z}_n = x_n \circledast y_n = \sum_{k=-\infty}^{\infty} z_{n-kN}$$

As we have seen in class, if we have two sequences of N samples, and we add $N-1$ zeroes, the circular convolution of these will be equivalent to the linear convolution[3] of the original sequences.

To calculate manually a circular convolution (length N), first compute the linear convolution (length $2N-1$) and then bring all the samples to the range between 0 and $N-1$. Divide by N and order the samples using the remainder.

5.6.6 Modulation

With the same x_n and y_n as before,

$$\text{DFT}(x_n \cdot y_n) = \frac{1}{N} \cdot X_k \circledast Y_k$$

Proof on page 111 of the book.

5.6.7 Parseval's relation

It can be shown that:

$$\sum_{n=0}^{N-1} x_n \cdot y_n^* = \frac{1}{N} \sum_{k=0}^{N-1} X_k \cdot Y_k^*$$

[3] The linear convolution of two sequences of N samples is a sequence of length $2N-1$.

For $x_n = y_n$, we obtain **Parseval's relation** for sequences of limited duration:

$$E_n = \sum_{n=0}^{N-1} |x_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X_k|^2$$

The **energy** of the sequence x_n can be obtained both in the time and frequency domain. The **power** of the signal x_n can be computed as:

$$P_n = \frac{1}{N} E_n = \frac{1}{N} \sum_{n=0}^{N-1} |x_n|^2 = \frac{1}{N^2} \sum_{k=0}^{N-1} |X_k|^2$$

For example, taking $\tilde{x}_n = A\cos(2\pi n/100)$ of $N = 100$, its DFT is

$$X_k = 100\frac{A}{2}(\delta_{k-1} + \delta_{k-99})$$

The power will be

$$P_n = \frac{1}{100^2} 2 \left(100\frac{A}{2}\right)^2 = \frac{A^2}{2}$$

5.6.8 Circular autocorrelation of a sequence

$$R_x[m] = IDFT(|X_k|^2) = \sum_{n=0}^{N-1} x_{n+m} x_n^* = x_m \circledast x_{-m}^*$$

All the properties we saw in the previous chapters are still valid.

5.6.9 Circular cross-correlation of a sequence

$$R_{yx}[m] = \sum_{n=0}^{N-1} y_{n+m} x_n^* = y_m \circledast x_{-m}^*$$

If $y_n = h_n \circledast x_n$, we have that $R_{yx}[m] = h_m \circledast R_x[m]$.

5.7 Applications of the DFT

5.7.1 Translation by a non-integer number of samples

This idea might not make much sense at first, but it is understandable when we remember that we get the sequence x_n sampling a continuous signal $x(t)$. Consider that we want to find the following sequence

$$y_n = x(t-\tau)|_{t=nT} = x(t)|_{t=nT-\tau}$$

given

$$x_n = x(t)|_{t=nT}$$

The relation of their Fourier transform is the following:

$$Y(f) = X(f)e^{-j2\pi f\tau} \qquad -\frac{1}{2T} < f < \frac{1}{2T}$$

Or, using normalized frequency:

$$Y(\phi) = X(\phi)e^{-j2\pi\phi\tau/T} \qquad -\frac{1}{2} < \phi < \frac{1}{2} \tag{5.1}$$

The relation between the DFTs X_k and Y_k can be found putting $\phi = \frac{k}{N}$ in the formula (5.1). However, we have to be careful, as the exponential from that formula is not periodic of period $\phi = 1$ as in the case of the translation by a integer number of samples (integer τ/T).

Equation (5.1) is in fact valid only in the interval $-\frac{1}{2} < \phi \leq \frac{1}{2}$, and then it repeats with step $\phi = 1$. So, we have to make the following distinction:

$$Y_k = \begin{cases} X_k e^{-j2\pi\frac{k}{N}\frac{\tau}{T}} & \text{for} \qquad 0 \leq k \leq \frac{N}{2} \\ X_k e^{-j2\pi\left(\frac{k}{N}-1\right)\frac{\tau}{T}} & \text{for} \quad \frac{N}{2}+1 \leq k \leq N-1 \end{cases}$$

y_n can be then found as $IDFT(Y_k)$.

5.7.2 Linear convolution

We have already talked about the relation between linear and circular convolutions. More information about this can be found on page 117 of the book.

5.7.3 Interpolation in frequency with zero-padding in time

We have seen that

$$X_k = X(f)|_{f=\frac{k}{NT}}$$

The frequencies $f = \frac{k}{NT}$ corresponding to the samples X_k get closer from each other as the duration $T_0 = TN$ of the sequence increases. The sampling interval in frequency is in fact:

$$\Delta f = \frac{1}{T_0} = \frac{1}{NT}$$

Once the temporal sampling interval has been fixed, the duration of the sequence can be increased adding P null samples to the right/left of the original sequence x_n (operation called *zero-padding in time*).

For example, if we want to reduce to half the sampling interval in f (*interpolation by a factor of 2*), it is enough to add N null samples to the sequence x_n, in order to double its duration. This way,

$$\Delta f_2 = \frac{1}{2NT} = \frac{1}{2T_0} = \frac{\Delta f}{2}$$

5.7.4 Interpolation in time with zero-padding in frequency

It is the dual operation of the previous one. If the samples x_n correspond to the time instants $t = \frac{nT_0}{N}$, the (bilateral) bandwidth of the DFT X_k is:

$$B = \frac{1}{\Delta t} = \frac{N}{T_0}$$

Once we fix the frequency sampling interval $\frac{1}{T_0}$, B can be increased adding P null samples to the sequence X_k (*zero-padding in frequency*).

For example, to reduce to half a sampling interval in t, it is enough to add N null samples to X_k in order to double the bandwidth B that the DFT occupies.

$$\Delta t_2 = \frac{T_0}{2N} = \frac{1}{2B} = \frac{\Delta t}{2}$$

Sequenza	**DFT**
x_n	X_k
y_n	Y_k
$ax_n + by_n$	$aX_k + bX_k$
circular x_{n-m}	$X_k e^{-j2\pi \frac{km}{N}}$
$e^{-k2\pi \frac{nm}{N}} x_n$	circular X_{k+m}
$\sum_{m=0}^{N-1} x_m y_{n-m}$ circular	$X_k Y_k$
$x_n y_n$	circular $\frac{1}{N} \sum_{m=0}^{N-1} X_m Y_{k-m}$
x_n^*	circular X_{-k}^*
circular x_{-n}^*	X_k^*

6 | Richiami di probabilità

We use probability to describe phenomenons which can be thought of as *experiments*, whose results vary among different N trials.

If we make N trials, and that N is high enough, the **relative frequency** of the results/events is close to their probability:

$$f_k = \frac{N_k}{N} \approx P(k)$$

The **histogram of the results** is a graph that plots the relative frequencies of each event.

6.1 Properties of probability

- The probability is a number between 0 and 1.

$$0 \leq P(A) \leq 1$$

- The set S of all the possible results is a *sure* result:

$$P(S) = 1$$

- Probability of the union (**or**) of two events:

$$P(A + B) = P(A) + P(B) - P(AB)$$

- Probability of the intersection (**and**) of two events:

$$P(AB) = P(A|B) \cdot P(B)$$

  where $P(A|B)$ is the **conditioned probability** of A given that B has occurred.

6.1.1 Bayes theorem

It is easy to show that

$$P(AB) = P(A|B) \cdot P(B)$$
$$P(AB) = P(B|A) \cdot P(A)$$

From those two, we get the expression of **Bayes theorem**:

$$\boxed{P(A|B) = P(B|A) \cdot \frac{P(A)}{P(B)}}$$

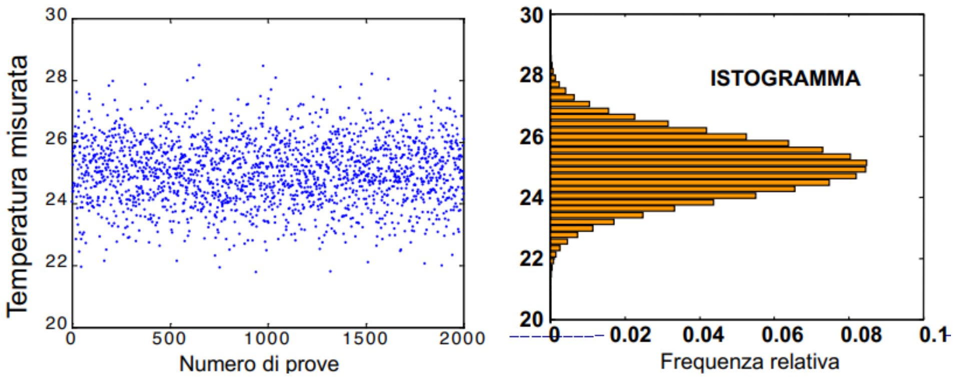
6.2 Discrete and continuous variables

A **random variable** is a real number[1] associated to the result of an experiment. If the possible results are numerable, the variable is said to be **discrete**. However, if the variable can take a continuous range of values (infinite results!) it is called a **continuous** random variable.

The concept of **relative frequency** appears again when approximating a continuous set of values with a finite number of small intervals (discretization). The random variable becomes discrete, and we can approximate the probability as a limit of the relative frequency for high N.

6.3 Histograms

Once we have turned a continuous random variable discrete, it is possible to plot its **histogram** as a graph of the relative frequencies of the results of each intervals on which the continuous set of the results have been divided.



Warning: the values of the *continuous* random variables (after making them discrete) depend on the dimension of the chosen intervals: as they get tighter, the values of the histogram become smaller.

6.4 Probability distribution

For a random variable x, the probability distribution $F_x(a)$ indicates the probability of getting an output equal or smaller than a:

$$\boxed{F_x(a) = P(x \leq a)}$$

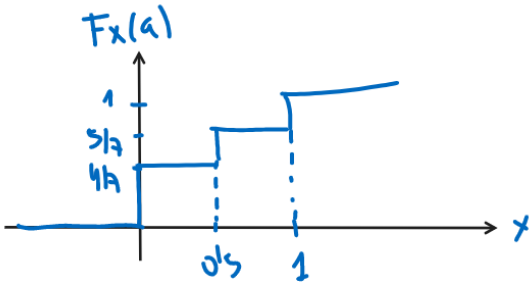
[1]If we refer to the possible outcomes of throwing a dice as a, b, c, d, e, f, we are **not** defining a random variable.

Properties:

1) It is monotonous and increasing.
2) $F_x(-\infty) = 0$.
3) $F_x(\infty) = 1$.

Example: traffic lights. Red: $x = 0$, yellow: $x = 0.5$, green $x = 1$.

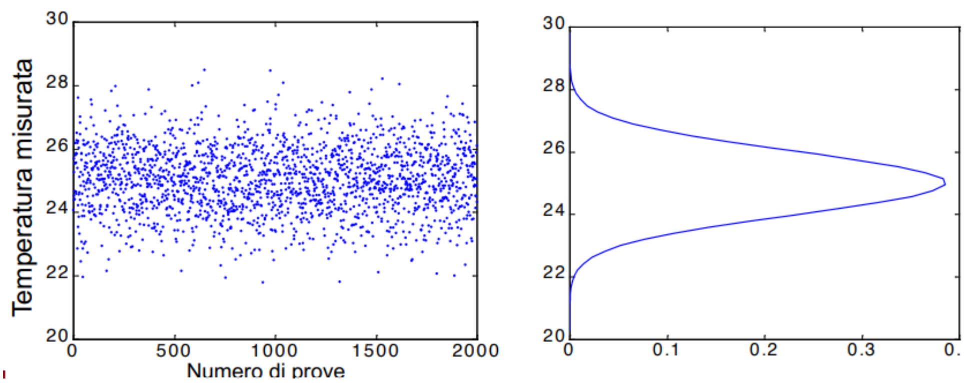
$$P(x = 0) = 4/7 \qquad P(x = 0.5) = 1/7 \qquad P(x = 1) = 2/7$$



6.5 Probability density

Adapting the idea of the histograms for continuous random variables, the concept of **probability density function**[2] (pdf) arises.

Small intervals must be used so that the probability distribution can be retained constant within them. Divide the histogram value by the size of the interval (so that the result is independent of the size of the interval). Use a very large number of trials (the higher the smaller the interval) so that relative frequencies and probabilities nearly coincide.



[2]In Italian, *densità di probabilità (ddp)*.

The **probability density** $p_x(a)$ of a continuous random variable can be defined as:

$$p_x(a) = \lim_{da \to 0} \frac{P(a < x \leq a + da)}{da} = \frac{d}{da} F_x(a)$$

Know that:

$$p_x(a) \geq 0 \qquad \int_{-\infty}^{\infty} p_x(a)\, da = 1$$

Note: the probability density is also written as $f_X(x)$, which is **not** the frequency in Hz. We will not use this notation.

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Know that:

$$\begin{aligned} P[x_1 \leq X \leq x_2] &= P[X \leq x_2] - P[X \leq x_1] \\ &= F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)\, dx \end{aligned}$$

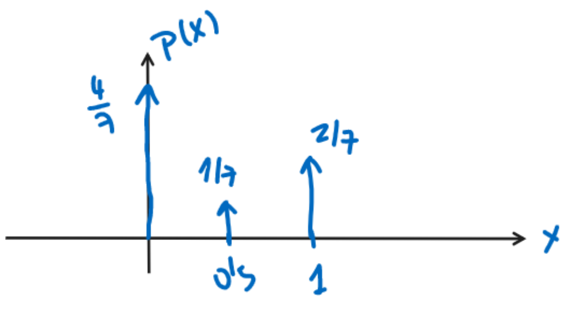
Keep in mind that the relation between probability and area under the probability density curve:

$$P(a_1 < x \leq a_2) = \int_{a_1}^{a_2} p(a)\, da$$

Important:

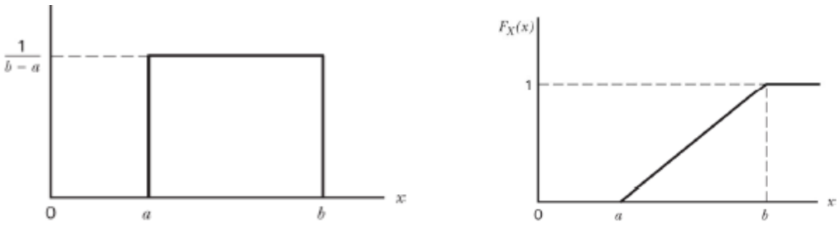
$$P(-\infty < x < \infty) = \int_{-\infty}^{\infty} p(a)\, da = 1$$

Continuing with the previous example of the traffic lights, its probability distribution:



Example: uniform/continuous distribution

$$p(x) = \begin{cases} 0 & x \leq a \\ \dfrac{1}{b-a} & a < x \leq b \\ 0 & x > b \end{cases} \qquad F(x) = \begin{cases} 0 & x \leq a \\ \dfrac{x-a}{b-a} & a < x \leq b \\ 1 & x > b \end{cases}$$



(a) $p(x)$ (b) $F(x)$

Figure 6.1: Uniform/continuous distribution

Check this results with the concepts that are explained in the following lines:

$$E[x] = \frac{b+a}{2}$$
$$E[x^2] = \frac{b^3 - a^3}{3(b-a)}$$
$$\sigma_x^2 = \frac{(b-a)^2}{12} = \frac{\Delta^2}{12}$$

6.6 Expectation value

In italian, *valor medio* m_x, or *valore atteso* $E[x]$, or **statistic moment of order one**. It is defined as:

$$\mu_x = m_x = E[x] = \sum_{k=1}^{N} a_k \cdot p(a_k)$$

For an experiment that is repeated an infinitely big amount of times (high N), m_x can be understood as the arithmetic mean of the results. **Baricenter** of the area under the curve of the probability density.

For example, if we take a dice:

$$m_x = E[x] = \sum_{k=1}^{6} k \cdot \frac{1}{6} = 3.5$$

The value of m_x does not need to be one of the possible outcomes of the experiment.

For a **continuous variable**, the same concept can be defined:

$$\mu_x = m_x = E[x] = \int_{-\infty}^{\infty} a \cdot p(a)\, da \approx \frac{1}{N} \sum_{i=1}^{N} x_i$$

If N is high enough, m_x can be taken as the arithmetic mean of the results. Other possible notation:

$$m_x = \mu_x = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

6.6.1 Properties of the mean value

- The mean value of a function $y = g(x)$ of a random variable x can be found using:

  $$E[g(x)] = \int_{-\infty}^{\infty} g(a)\, p_x(a)\, da$$

  The result is analogous for a function of more than one random variables. Note that for high N,

  $$E[y] \approx \frac{1}{N}\sum_{i=1}^{N} y_i = \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

  Other possible notation: $Y = g(X)$. If the pdf is $f_X(x)$,

  $$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy \quad \longrightarrow \quad E[g(X)] = \int_{-\infty}^{\infty} g(x) f_x(x) dx$$

- The mean value of a linear combination of N random variables is the linear combination of the mean values:

  $$E\left[\sum_{n=1}^{N} b_n x_n\right] = \sum_{n=1}^{N} E[b_n x_n] = \sum_{n=1}^{N} b_n E[x_n]$$

6.7 Statistical momentums

The **statistic moment of order n** of a random variable x is:

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

The expectation value is the momentum of order 1. **Central momentum of order n**:

$$E[(X - \mu_X)^n] = \int_{-\infty}^{\infty} (x - \mu_X)^n f_X(x) dx$$

6.7.1 Quadratic mean value

Also called *statistic power* or *statistical moment of order 2*, the **quadratic mean value** $E[x^2]$ can be interpreted as the arithmetic mean of the square of a lot of samples:

$$E[x^2] = \int_{-\infty}^{\infty} a^2 p(a)\, da \approx \frac{1}{N}\sum_{i=1}^{N} x_i^2$$

6.7.2 Variance

The **variance** σ_x^2 of a random variable x, also known as *central moment of order 2*, is the quadratic mean value of the difference between x and its mean value m_x:

$$\sigma_x^2 = E[(x - m_x)^2] = E[x^2] + m_x^2 - 2E[x]m_x = E[x^2] - m_x^2$$

The variance is the difference between the quadratic mean value and the square of the mean value:

$$\boxed{\sigma_x^2 = E[(x - m_x)^2] = E[x^2] - m_x^2}$$

Note that if the mean value is null, the variance is equal to the quadratic mean value.

6.7.3 Standard deviation

The square root of the variance is called **standard deviation** of the random variable x:

$$\boxed{\sigma_x = \sqrt{\sigma_x^2}}$$

This number measures the *dispersion* of the samples with respect to the mean value of x. The higher σ_x is, the more sparse and far will be the results from the mean value.

Gaussian probability

A Gaussian random variable has Gaussian probability density:

$$\boxed{p(a) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(a - m_x)^2}{2\sigma_x^2}\right)}$$

Limits:

$$P\left(m_x - \sigma_x < x \leq m_x + \sigma_x\right) = \int_{m_x-\sigma_x}^{m_x+\sigma_x} \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(a - m_x)^2}{2\sigma_x^2}\right) da \approx 0.683$$

$$P\left(m_x - 2\sigma_x < x \leq m_x + 2\sigma_x\right) = \int_{m_x-2\sigma_x}^{m_x+2\sigma_x} \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(a - m_x)^2}{2\sigma_x^2}\right) da \approx 0.954$$

$$P\left(m_x - 3\sigma_x < x \leq m_x + 3\sigma_x\right) = \int_{m_x-3\sigma_x}^{m_x+3\sigma_x} \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(a - m_x)^2}{2\sigma_x^2}\right) da \approx 0.997$$

6.8 Joint probability distributions

The **joint probability distribution** of two random variables x and w is represented as $p_{x,w}(a, b)$, and it indicates the probability to have $x = a$ and $w = b$ at the same time.

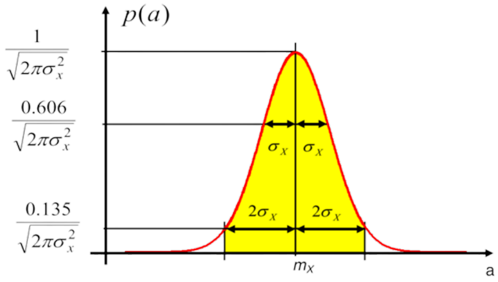


Figure 6.2: Gaussian probability density.

For example (fig. 6.3), take a system formed by two dices. x and w take values from 1 to 6, so we will have 36 pairs of equiprobable results: $(1,1),(1,2),...(2,1),...,(6,6)$. Therefore,

$$p_{x,w}(a,b) = \frac{1}{36}$$

In this example it is possible to see that the joint probability can be written as the product of the pdfs of each random variables. When this happens, x and w are **statistically independent random variables.**

$$\boxed{p_{x,w}(a,b) = p_x(a)p_w(b)}$$

This means that the value taken by one of the variables does **not** affect in any way the result of the other one.

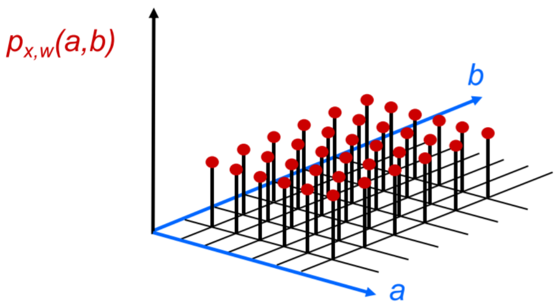


Figure 6.3: Joint probability density of our example with the dices.

6.9 Dependent random variables

We will have conditioned probability densities. Using Bayes formula, we have that:

$$\boxed{p_{x,y}(a,c) = p_{y|x=a}(c|x=a)p_x(a)}$$

The symbol | means *given that.*

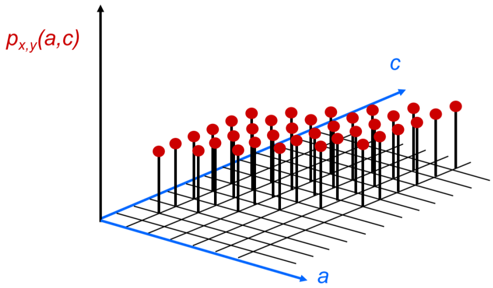


Figure 6.4: Consider the random variables x and w that describe the output of two dices. We want to compute $p_{x,y}(a,c)$ of the random variables x and $y = x + w$. First, notice that y depends on the value of x. When $x = a$, y can take values $\{a+1, a+2, a+3, a+4, a+5, a+6\}$ with probability 1/6. This can be written as: $P_{y|x=a}(c|x = a) = \frac{1}{6}$ for $1 + a \leq c \leq 6 + a$.

See that:

$$\sum_c p_{x,y}(a,c) = p_x(a)$$
$$\sum_a p_{x,y}(a,c) = p_y(b)$$

If the variables are independent,

$$p_{y|x=a}(c|x = a) = p_y(c)$$

and we recover the previous formula

$$p_{x,y}(a,c) = p_y(c)p_x(a)$$

If our random variables x and y are **continuous**, their joint probability density is

$$p_{x,y}(a,b) = \frac{P(a < x < a + da, b < y < b + db)}{da\, db}$$

The function $p_{x,y}(a,b)$ must fulfill the following properties:

$$p_{x,y}(a,b) \geq 0 \qquad \iint p_{x,y}(a,b) da\, db = 1$$
$$\int_{-\infty}^{\infty} p_{x,y}(a,b)\, db = p_x(a)$$

In fact, integrating $p_{x,y}(a,b)\, db$ we are summing the probabilities of all the possible joint elementary results that can be obtained with $a < x < a + da$.

Note that if:

$$p_{x,y}(a,b) = p_{y|x=a}(b|x = a)p_x(a) \quad \rightarrow \quad p_{y|x=a}(b|x = a) = \frac{p_{x,y}(a,b)}{p_x(a)}$$

The following integral must be unitary:

$$\int p_{y|x=a}(b|x=a)\,db = \frac{1}{p_x(a)}\int p_{x,y}(a,b)\,db = 1$$

Remember that independence of the random variables implies that $p_{y|x=a}(b|x=a) = p_y(b)$.

6.10 Sum of random variables

Let x and y be two **independent** random variables. The probability density function of random variable $z = x + y$ will be the **convolution** of the pdf of x and y:

$$p_z = p_x * p_y$$

Central limit theorem: The **sum** of a big number N of independent random variables x_i has a pdf close to a **Gaussian**, independently of the individual pdfs:

$$p_y(a) = p_{x_1}(a) * p_{x_2}(a) * ... * p_{x_N}(a) \approx \frac{1}{\sqrt{2\pi\sigma_y^2}}\exp\left(-\frac{(a-m_y)^2}{2\sigma_y^2}\right)$$

For $N \sim 5$ the approximation is already good.

The **variance** of the sum will be equal to the sum of variances:

$$\sigma_z^2 = \sum_i \sigma_i^2$$

6.11 Covariance

Given two random variables X and Y, the **covariance** is the difference between the expectation value of the product of both variables and the product of the expectation values:

$$\text{cov}[XY] = E[(X - E[X])(Y - E[Y])] = E[XY] - \mu_x\mu_y$$

6.11.1 Correlation coefficient

$$\rho = \frac{\text{cov}[XY]}{\sigma_X\sigma_Y}$$

where $\sigma_X\sigma_Y$ is the product of the standard deviations.

6.11.2 Uncorrelated variables

Two random variables x and y are **uncorrelated** if and only if

$$\boxed{\text{cov}[xy] = 0}$$

and thus, $E[xy] = E[x]E[y]$. Indeed, for statistically independent random variables, we know that

$$E[xy] = E[x]E[y]$$

as

$$\begin{aligned} E[xy] &= \iint xyf_{X,Y}(x,y)dxdy = \iint xyf_X(x)f_Y(y)dxdy \\ &= \int xf_X(x)dx \int yf_Y(y)dy = E[x]E[y] \end{aligned}$$

Analogously, if x and y are independent random variables and $f(x)$ and $g(y)$ are arbitrary functions,

$$E[f(x)g(y)] = E[f(x)]E[g(x)]$$

Be careful (*typical question for oral exams*):

- Two random variables can be **uncorrelated** even if they are **not independent**.
- Two **independent** random variables are always **uncorrelated**.

So, being uncorrelated is necessary but **not sufficient** condition for two random variables to be independent.

6.11.3 Orthogonal variables

Two random variables are said to be **orthogonal** if and only if

$$E[xy] = 0$$

This is not *stronger* than being uncorrelated.

7 | Processi casuali: parte I

Let's start this chapter by doing a small summary of what we have learned up to this point.

7.0.1 Random variables

$$\mu_x = m_x = E[x] = \int_{-\infty}^{\infty} a\, p(a)\, da = \int_{-\infty}^{\infty} x f_X(x) dx$$
$$E[x^2] = \int_{-\infty}^{\infty} a^2\, p(a)\, da$$
$$\sigma_x^2 = E[(x-m_x)^2] = E[x^2] + m_x^2 - 2E[x]m_x = E[x^2] - m_x^2$$

For two random variables x and y:

$$\text{cov}[XY] = E[(X - E[X])(Y - E[Y])] = E[XY] - \mu_x \mu_y$$
$$\rho = \frac{\text{cov}[XY]}{\sigma_X \sigma_Y}$$

If the variables are statistically **independent**:

$$p_{x,w}(a,b) = p_x(a) p_w(b)$$

and consequently

$$E[xy] = m_x m_y$$

If the variables are **uncorrelated**:

$$\text{cov}[xy] = 0 \qquad \rho = 0$$

Statistically independent $\Rightarrow$ uncorrelated, but uncorrelated $\not\Rightarrow$ statistically independent.

7.0.2 Gaussian probability density

$$\boxed{p(a) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(a-m_x)^2}{2\sigma_x^2}\right)}$$

Limits:

$$P\left(m_x - \sigma_x < x \leq m_x + \sigma_x\right) = \int_{m_x-\sigma_x}^{m_x+\sigma_x} \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(a-m_x)^2}{2\sigma_x^2}\right) da \approx 0.683$$
$$P\left(m_x - 2\sigma_x < x \leq m_x + 2\sigma_x\right) = \int_{m_x-2\sigma_x}^{m_x+2\sigma_x} \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(a-m_x)^2}{2\sigma_x^2}\right) da \approx 0.954$$
$$P\left(m_x - 3\sigma_x < x \leq m_x + 3\sigma_x\right) = \int_{m_x-3\sigma_x}^{m_x+3\sigma_x} \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(a-m_x)^2}{2\sigma_x^2}\right) da \approx 0.997$$

Example 1: on a previous lesson we considered a variable φ that was uniformly distributed between $-\pi$ and π:

$$f_\varphi = \frac{1}{2\pi}$$

If we take a RV $g(\varphi) = \cos(\varphi)$, then

$$E[g(\varphi)] = \int_{-\infty}^{\infty} g(\varphi) f_\varphi g(\varphi)\, d\varphi$$

Now let's take a signal that could be an electromagnetic wave

$$x(t) = \cos(2\pi f_0 t + \varphi) = \cos\left(2\pi f_0 (t + \frac{\varphi}{2\pi f_0})\right)$$

See that in this case $-1 \leq x(t=0) \leq 1$, with $E[x(t=0)] = 0$. Same for $E[x(t = \frac{1}{2\varphi})] = 0$, independently of the time. Using some math:

$$\begin{aligned} E[x(t)] &= E[\cos(2\pi f_0 t + \varphi)] = E[\cos(2\pi f_0 t)\cos\varphi - \sin(2\pi f_0 t)\sin\varphi] \\ &= \cos(2\pi f_0 t)\, E[\cos\varphi] - \sin(2\pi f_0 t)\, E[\sin\varphi] \\ &= \cos(2\pi f_0 t) \int_{-\pi}^{\pi} \cos\varphi \frac{1}{2\pi}\, d\varphi - \sin(2\pi f_0 t) \int_{-\pi}^{\pi} \sin\varphi \frac{1}{2\pi}\, d\varphi = 0 \end{aligned}$$

Now, we want to see if $x(t)$ is uniformly distributed too. If it was, remember that we could express its variance as:

$$\sigma^2 = \frac{\Delta^2}{12}$$

And taking $\Delta = 2$, $\sigma^2 = \frac{1}{3}$. Let's calculate it ourselves and see if it matches (spoiler: no).

$$\sigma_x^2 = E[x^2(t)] - \cancel{E^2[x(t)]} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2(2\pi f_0 t + \varphi)\, d\varphi \overset{t=0}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2(\varphi) d\varphi = \frac{1}{2\pi}\pi = \frac{1}{2}$$

The variable is more disperse than what we expected, so $x(t)$ is **not** uniformly distributed.

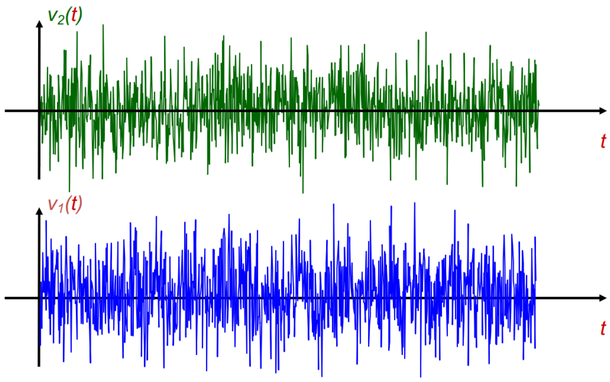
7.1 Deterministic signals

A signal is **deterministic** if at a given time t_0 it has associated a precise value $x(t_0)$. All the signals we have used until now are deterministic. However, a big part of the signals we find in practice cannot be represented in this way; they have associated certain value, but they are not repeatable. An example of this is the **thermal noise**.

7.2 Random processes

The thermal noise can be represented as the voltage $v(t)$ existing in a resistor, caused by the chaotic movement of the electrons as a consequence of having certain temperature ($> 0K$).

If we take two identical resistors and we measure the deterministic signals $v_1(t)$ and $v_2(t)$, we will obtain two signals that even if they have similar characteristics, they are different between them.



If our aim is to determine the effect of the thermal noise on a system, it is not useful at all to know deterministically the behavior of $v_1(t)$ along the first resistor if we then are going to use the second one.

Nevertheless, the smart thing to do is to describe the characteristics of the thermal noise which are common to all resistors of the same type and temperature. This way, we will be able to give(for example) the probability of certain values of the tension or the value of the power for any resistor.

Consequently, we abandon the concept of **certainty** of deterministic signals to take the **uncertainty**, described by the probability theory and characteristic of random processes.

A random process is a set of all the deterministic signals (*realizations of the process*) generated by the same source but independent between them. The value of the different realizations at time $t = t_k$ will be a random variable $x(t_k)$.

7.3 Stationary random processes

A random process is said to be **stationary** if its statistic characteristics do not depend on t. For instance, the probability density of the process is:

$$p(x_{t_i+t}) = p(x_{t_i}) \quad \text{for all values of } t$$

For a **stationary** random process the n-th order momentum is independent of time:

$$E[X_{t_i}^n] = \int_{-\infty}^{\infty} x_{t_i}^n p(x_{t_i})\, dx_{t_i}$$

Particularly the mean value and the variance are constant in time: $\mu_X(t) = \mu_X$ and $\sigma_X^2(t) = \sigma_X^2$.

7.3.1 Autocorrelation

$$R_X(t_1, t_2) = E[X_{t_1} X_{t_2}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_{t_1} x_{t_2} p(x_{t_1}, x_{t_2})\, dx_{t_1} dx_{t_2}$$

The autocorrelation measures how much the value taken by the realization of the process at time t_1 is related to the value of the same realization at time t_2.

If the random process is complex, the correct expression is:

$$R_X(t_1, t_2) = E[X(t_1) X^*(t_2)]$$

If we limit our analysis to **stationary** random processes, R_X does not depend on the instants t_1 and t_2 but only on the delay τ between the measurements:

$$E[X_{t_1}, X_{t_2}] = R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$$

The autocorrelation tells us how much the value taken by a realization of the random process at time $t + \tau$ is related to the value the same realization takes at time t.

The autocorrelation function depends on the joint probability density of $x(t)$ and $x(t + \tau)$.

- If the two random variables are *independent*, the joint pdf is equal to the product of the pdfs and the autocorrelation function coincides with the *square of the mean value* of the random process.
- If the random variable $x(t+\tau)$ depends on the value taken by $x(t)$, the autocorrelation function will have a different value to the square of the mean value of the random process.

Know that if we sample a continuous process $(m_x, \sigma_x^2, R_x(\tau))$ with period T, the discrete process will have equal m_x and σ_x^2 but the autocorrelation function will no longer be continuous:

$$R_x[n] = R_x(nT)$$

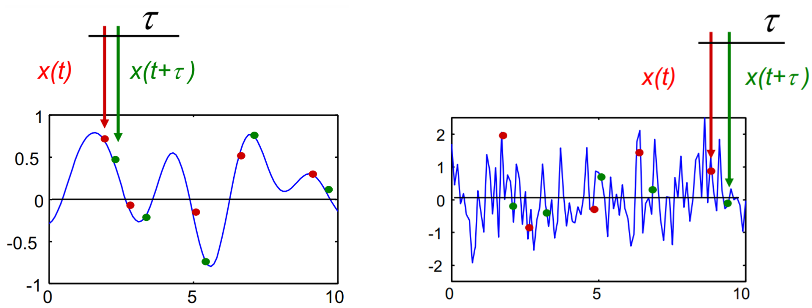
7.3.2 Slowly varying processes

If we have a high number N of trials, the following approximation is good:

$$R_x(\tau) = E[x(t)x(t+\tau)] \approx \frac{1}{N}\sum_{i=1}^{N} x_i(t)x_i(t+\tau)$$

Now let's consider a stationary random process with null mean value, with realizations that vary slowly in time.

If $x(t)$ evolves slowly in time with respect to a fixed value τ, $x(t+\tau)$ is similar to $x(t)$. The product $x_i(t)x_i(t+\tau)$ has often positive sign, in the various realizations and times t.



(a) Slowly varying process. (b) Rapidly varying process.

Figure 7.1: Autocorrelation of different kind of processes.

7.3.3 Rapidly varying processes

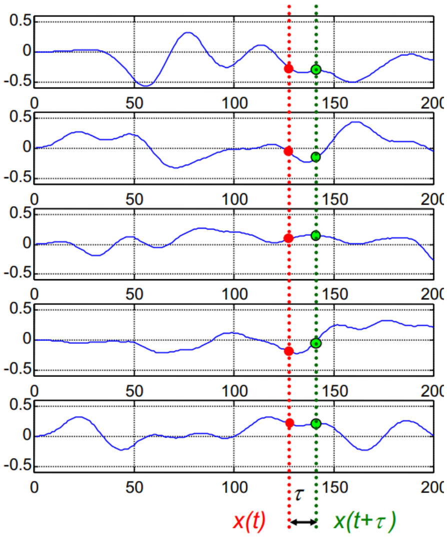
Consider a stationary random process with null mean value, with realizations that vary rapidly in time.

If $x(t)$ evolves rapidly in time with respect to the fixed value τ, $x(t+\tau)$ changes a lot from $x(t)$. The product $x_i(t)x_i(t+\tau)$ will have random sign in the different realizations and times t. From this we deduce that the autocorrelation will take a value close to zero, as we are summing practically the same amount of positive and negative terms.

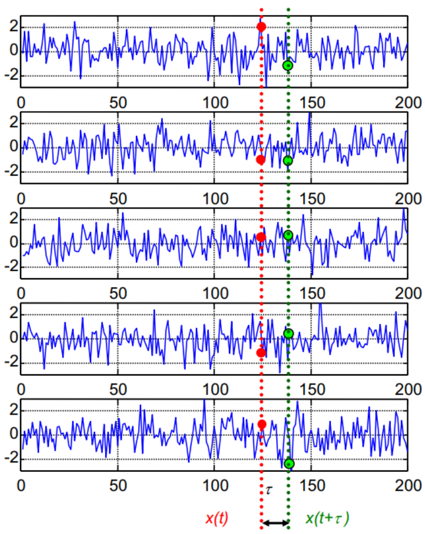
7.3.4 Properties of autocorrelation

1) $R_X(\tau) = E[X(t+\tau)X(t)]$ for all t.
2) Mean quadratic value: $R_X(0) = E[X^2(t)]$.
3) Even function: $R_X(\tau) = R_X(-\tau)$. We see this showing that:

$$E[x(t+\tau)x(t)] = E[x(t-\tau)x(t)]$$



(a) Slowly varying process.



(b) Rapidly varying process.

Figure 7.2: Autocorrelation of different kind of processes.

4) $|R_X(\tau)| \leq R_X(0)$. For this, take $x(t+\tau) \pm x(t)$:

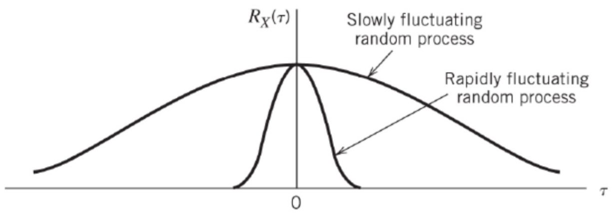
$$E[(x(t+\tau) \pm x(t))^2] \geq 0$$

$$E[x^2(t+\tau) + x^2(t) \pm 2x(t+\tau)x(t)] = E[x^2(t+\tau)] + E[x^2(t)] \pm 2E[x(t)x(t+\tau)]$$

We know that $E[x^2(t)] = R_X(0)$ but given that the process is stationary, we also know that $E[x^2(t+\tau)] = R_X(0)$. So,

$$E[(x(t+\tau) \pm x(t))^2] = R_X(0) + R_X(0) + 2R_X(\tau) \geq 0$$

$$R_X(0) \pm R_X(\tau) \geq 0 \quad \longrightarrow \quad R_X(\tau) \leq R_X(0),\ R_X(\tau) \geq -R_X(0)$$



7.4 Autocovariance

$$C_X(t_1, t_2) = E[(X(t_1) - \mu_X)(X(t_2) - \mu_X)] = R_X(t_2 - t_1) - \mu_X^2$$

The **autocovariance** function is defined as the square of the mean value of a random process subtracted to the autocorrelation function.

$$C_X(\tau = 0) = R_X(0) - \mu_X^2 = E[X(t)^2] - \mu_X^2 = \sigma_X^2$$

In the origin the autocovariance is equal to the variance of X.

7.5 Correlation coefficient

$$\rho_X(\tau) = \frac{C_X(\tau)}{C_X(0)} = \frac{C_X(\tau)}{\sigma_X^2}$$

It takes a value in the interval $[0,1]$, and it indicates how related are two points of the same random process.

- If both variables are *independent* or *uncorrelated*, the autocovariance function is null and so is the correlation coefficient.
- However, if the random variable $x(t+\tau)$ depends of the absolute value of $x(t)$, the modulus of the correlation function will have a value which as it gets closer to 1, it indicates a bigger dependence between the two random variables.

$$R_X(\tau) = E[x(t)x(t+\tau)] \approx \frac{1}{N}\sum_{i=1}^{N} x_i(t)x_i(t+\tau)$$

The autocorrelation evaluated in $\tau = 0$ is equal to the mean power of the random process.

$R_X(0)$ is the maximum value that the autocorrelation can take.

In a process on which the mean value is null, the correlation coefficient becomes the normalized autocorrelation:

$$\rho_X(\tau) = \frac{R_X(\tau)}{R_X(0)} = \frac{E[x(t)x(t+\tau)]}{E[x^2(t)]}$$

Example: sinusoidal signal[1] with random phase:

$$X(t) = A\cos(2\pi f_c t + \Theta) \qquad f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \le \theta \le \pi \\ 0, & \text{else} \end{cases}$$

We see that the phase is a uniformly distributed random variable. To calculate the autocovariance, remember that $C_X(\tau) = R_X(\tau) - \mu_X^2$. We have seen that for a function like this $\mu_X = 0$. So, $C_X(\tau) = R_X(\tau)$:

$$\begin{aligned} R_X(\tau) &= \mathbf{E}[X(t+\tau)X(t)] \\ &= \mathbf{E}\left[\frac{A^2}{2}\cos\left(4\pi f_c t + 2\pi f_c \tau + 2\Theta\right)\right] + \frac{A^2}{2}\mathbf{E}\left[\cos\left(2\pi f_c \tau\right)\right] \\ &= \frac{A^2}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos\left(4\pi f_c t + 2\pi f_c \tau + 2\theta\right)d\theta + \frac{A^2}{2}\cos\left(2\pi f_c \tau\right) = \\ &= \frac{A^2}{2}\cos(2\pi f_c \tau) \end{aligned}$$

[1] Later in the course we will see transportation of signals. It is the reason for the c (*carrier*) in f_c.

We can check that the solution fulfills the properties we have seen, as a way to confirm the validity of our calculation.

Take $\tau = \frac{1}{4f_c}$.

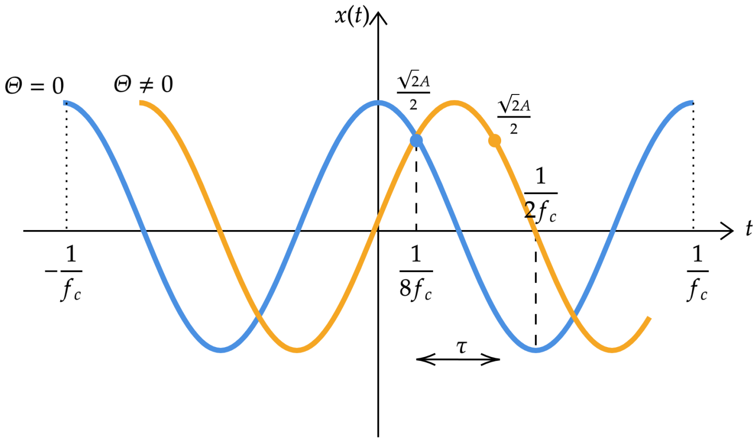
$$R_X\left(\tau = \frac{1}{4f_c}\right) = \frac{A^2}{2}\cos\left(2\pi f_c \frac{1}{4f_c}\right) = 0$$

Consequently,

$$R_X\left(\tau = \frac{1}{4f_c}\right) = C_X\left(\tau = \frac{1}{4f_c}\right) = 0 \qquad \rho_X\left(\tau = \frac{1}{4f_c}\right) = \frac{C_X(\tau)^2}{\sigma_X^2} = 0$$

Take $x(t_1)$ and $x(t_2)$. We measure $x(t_1) = \frac{\sqrt{2}}{2}A$, but we do not know Θ, so we can not know if that value is given by the blue or by the orange line.

We can say that $x(t_2) = \pm\frac{\sqrt{2}}{2}A$, each with probability of 50%, but we can not say which of them is the true value. We can **not** predict what happens in t_2. That is the reason why the correlation coefficient is null.



Are the two variables uncorrelated or are they statistically independent?

$$p_x\left(x(t_1)\middle| x(t_2) = \frac{A\sqrt{2}}{2}\right) = \frac{1}{2}\delta\left(x - \frac{A\sqrt{2}}{2}\right) + \frac{1}{2}\delta\left(x + \frac{A\sqrt{2}}{2}\right)$$

The 1/2 stands for the 50% probability.

Are they uncorrelated or statistically independent variables? They are **uncorrelated**. See that the conditioned probability distribution is really *conditioned*, it is not the product of the individual pdfs. Being independent is sufficient to be uncorrelated, but it is not necessary for it. *Independent* is stronger than *uncorrelated*.

Independence	**Uncorrelation**
$C_X(\tau) = 0 \Rightarrow \rho_X(\tau) = 0$	$C_X(\tau) = 0 \Rightarrow \rho_X(\tau) = 0$
$R_X(\tau) = \mu_X^2$	$R_X(\tau) = \mu_X^2$
$p_{x,w}(a,b) = p_x(a)p_w(b)$	$p_{x,y}(a,c) = p_{y\|x=a}(c\|x=a)p_x(a)$

The **correlation coefficient** of a process is a function whose values are limited between -1 and 1. Obviously **its value in $\tau = 0$ is unitary** and in most of the cases, the only maximum.

The value of the correlation coefficient as a function of τ is a measure of the predictability of a realization of the process at time $t + \tau$ knowing the value of the realization at time t.

The closer the value of the correlation coefficient in τ is to 1 or -1, the more precise will be the prediction of the value that the realization of the process will take at time $t + \tau$, knowing the value at t.

7.6 Relation between correlation and predictability

For briefness, we will write the random variables $x(t_1)$ and (x_2) as x_1 and x_2. It can be shown that:

- The more correlated the variables x_1 and x_2 are (with null mean value), the better we can estimate the value of one of them using the other.
- The best estimation will be achieved when the estimation error is uncorrelated with the data.

Saying that the random variable x_2 is correlated with the random variable x_1 is equivalent to say that the value taken by x_2 is in part proportional to the one x_1 takes, plus a random variable independent of x_1 (and therefore non-predictable):

$$x_2 = rx_1 + n$$

The stationarity of a process ensures that the variance σ^2 of both x_1 and x_2 is the same, imposing a relation between the variance of n and r. We have that $m_x = 0$ and:

$$E[x_1^2] = E[x_2^2] = \sigma^2$$

From which:

$$E[x_2^2] = E[(rx_1 + n)^2] = \sigma^2$$
$$r^2 E[x_1^2] + E[n^2] + 2rE[x_1 n] = \sigma^2$$
$$r^2\sigma^2 + E[n^2] = \sigma^2$$

Using that $E[x_1 n] = E[x_1]E[n]$ and $E[x_1] = 0$, we obtain the following expression:

$$E[n^2] = (1 - r^2)\sigma^2$$

The closer r is to 1, the smaller the mean quadratic value (and consequently the variance) or n is: x_2 will deviate less from the value of x_1.

From the definition of the correlation coefficient is easy to deduce that:

$$r = \frac{E[x_1 x_2]}{E[x_1^2]} = \rho(\tau)$$

The prediction of the future behavior of the random process (at time t_2) made with its current value (at time t_1), gets more precise as the correlation coefficient gets closer to 1.

7.6.1 Linear estimation of x_2 using x_1

Knowing the current value that x_1 takes, the aim is to predict (estimate) in the possible way the value that x_2 will take, finding the proportionality coefficient a. The **linear estimation** will be:

$$\hat{x_2} = ax_1$$

The optimum value of a is obtained when the error (in quadratic mean) of the estimation is minimum, this is, if the difference (in quadratic mean) of the estimated value and the effective value taken by x_2 is minimum. We need to minimize:

$$E[(x_2 - \hat{x}_2)^2] = E[(x_2 - ax_1)^2]$$

Taking the derivative with respect to a and making it equal to zero, the optimal value of a is obtained:

$$a = \frac{E[x_1 x_2]}{E[x_1^2]} = r = \rho(\tau)$$

The optimal value of a coincides, obviously, with the coefficient r.

Knowing this we can define a procedure to estimate the value that x_2 will take once we know the value taken by x_1.

1) **Learning:** analyzing N (big number) realizations of the random process, the correlation coefficient is computed:

$$\rho_x(\tau) \approx \frac{\sum_{i=1}^{N} x_i(t)x_i(t+\tau)}{\sum_{i=1}^{N} x_i^2(t)}$$

2) **Prediction:** $x(t+\tau)$ is estimated using $x(t)$:

$$\hat{x}(t+\tau) = \rho(\tau)x(t)$$

7.6.2 Estimation error

Note that the estimation error, caused only by the random variable n, has equal mean quadratic value to the one of n:

$$E[(x_2 - \hat{x}_2)^2] = E[(x_2 - ax_1)^2] = \sigma^2(1 - r^2)$$

From this expression we immediately understand that:

1) The estimation error is 0 if the random variables are totally correlated ($|r| = 1$).
2) The estimation error is maximum if the random variables are uncorrelated ($|r| = 0$).

In the end, it is important to note that the estimation error is uncorrelated with the datum x_1. In fact, if we compute the covariance:

$$E[x_1(x_2 - ax_1)] = E[x_1x_2 - ax_1^2] = E[x_1x_2] - aE[x_1^2] = rE[x_1^2] - rE[x_1^2] = 0$$

8 | Processi casuali: parte II

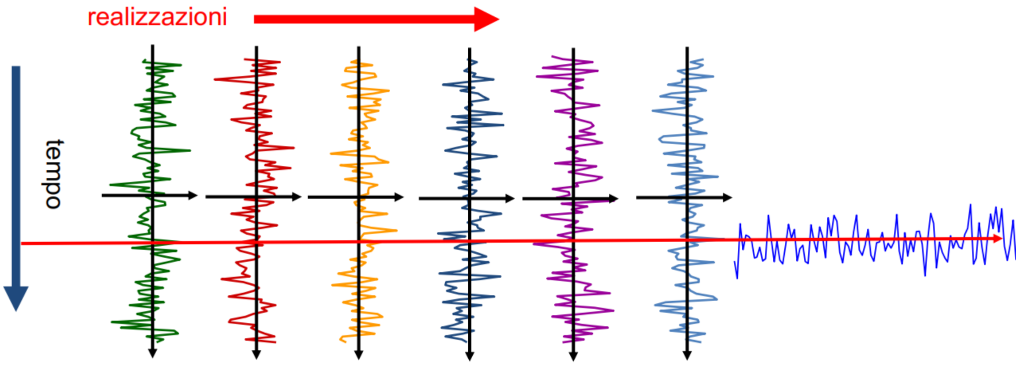
Independence implies uncorrelation, but uncorrelation does not imply independence. The only exception is if the process is Gaussian.

8.1 Gaussian stationary processes

If the process is **Gaussian** and the variables t_1, t_2, t_n are **uncorrelated**, they are also **statistically independent**.

8.2 Ergodic processes

Among the **stationary** random processes, there are some for which all the statistic properties can be obtained for a single realization. These are called **ergodic processes**.



All the realizations taken on a known instant give the same statistic information that can be obtained from a long-time observation of a single process. Obviously, for a process to be ergodic, it must be stationary.

8.2.1 Ergodic for the mean

A random process is **ergodic for the mean** if the time mean value of a single realization is equal to the statistical mean value of the process measured for every value of t.

$$\mu_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt \quad \text{temporal mean}$$
$$m_X = E[X] = \int_{-\infty}^{\infty} x f_X(x)dx \quad \text{statistical mean}$$

An ergodic process **must** be stationary, as the statistic means can not be functions of time. The opposite is not necessarily true.

8.2.2 Ergodic for the autocorrelation

A random process is **ergodic for the autocorrelation** if the autocorrelation of the set (the statistical one) is equal to the temporal autocorrelation.

$$R_X(\tau) = E[X(t+\tau)X^*(t)] = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t+\tau)x^*(t)dt$$

8.2.3 Ergodic process

All the statistical characteristics can be derived from a *single realization in time*. The time mean value and the autocorrelation estimated in time are equal to the ones of the set.

A **sufficient condition** for a random process to be ergodic is that the correlation coefficient goes to zero in a finite time (τ_0). At distance τ_0 the samples are uncorrelated. The temporal averages of a big amount N of samples (in time, always for time $> \tau_0$) are equal to the statistical mean values.

$$m_X = E\left[X_n\right] <= \frac{1}{N}\sum_0^{N-1} X_n$$
$$E\left[X^2{}_n\right] <= \frac{1}{N}\sum_0^{N-1} X^2{}_n$$
$$R_X[m] <= \frac{1}{N}\sum_0^{N-1} X_{n+m}X_n$$

8.3 Power spectrum

Wiener Knitchine's theorem: the Fourier transform of the set autocorrelation function $R_X(\tau)$ of a random stationary process is the **power spectrum of the signal** (or *spectral power density*) $S_X(f)$:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau$$
$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\exp(j2\pi f\tau)df$$

The autocorrelation in $\tau = 0$ $R_X(0)$ is equal to the **mean power of the random process**:

$$R_X(0) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x^*(t)dt = P_X$$

Therefore we have that:

$$\int_{-\infty}^{\infty} S_X(f)df = R_X(0) = P_X$$

The spectral power density is real due to the properties of the Fourier transform.

8.3.1 Properties

$$\begin{aligned}
&S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)d\tau \\
&S_X(f) \geq 0 \text{ for all } f \\
&S_X(-f) = S_X(f) \\
&E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f)df = R_X(0) = P
\end{aligned}$$

The last property shows that there are three different ways to compute the **power of the process**. This is the typical theory question: *how to compute the power of the process.*

Example: take a random process $X(t)$ and a sinusoidal signal which is a function of a random phase, uniformly distributed between 0 and 2π. $X(t)$ is statistically independent of the phase Θ.

$$Y(t) = X(t)\cos(2\pi f_c t + \Theta)$$

The autocorrelation of Y is:

$$\begin{aligned}
R_Y(\tau) &= \mathbf{E}[Y(t+\tau)Y(t)] \\
&= \mathbf{E}\left[X(t+\tau)\cos\left(2\pi f_c t + 2\pi f_c \tau + \Theta\right)X(t)\cos\left(2\pi f_c t + \Theta\right)\right] \\
&= \mathbf{E}[X(t+\tau)X(t)]\mathbf{E}\left[\cos\left(2\pi f_c t + 2\pi f_c \tau + \Theta\right)\cos\left(2\pi f_c t + \Theta\right)\right] \\
&= \frac{1}{2}R_X(\tau)\mathbf{E}\left[\cos\left(2\pi f_c \tau\right) + \cos\left(4\pi f_c t + 2\pi f_c \tau + 2\Theta\right)\right] \\
&= \frac{1}{2}R_X(\tau)\cos(2\pi f_c \tau)
\end{aligned}$$

Were we have used the statistical independence of $X(t)$ and the phase. Taking its Fourier transform:

$$S_Y(f) = \frac{1}{4}[S_X(f - f_c) + S_X(f + f_c)]$$

8.4 Continuous white processes

If the process $x(t)$ is **white**, its autocorrelation function is null everywhere except in the origin $\tau = 0$ (impulsive!) and its frequency spectrum is constant.

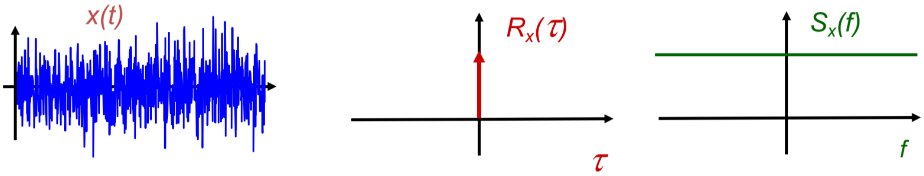


Figure 8.1: Continuous white process.

All the components of the frequency are present in the spectrum and that is the reason why it is called **white**. It is an analogy with white light, that contains all the visible wavelengths.

$R_X(\tau)$ can have a constant term, so rigorously speaking, $C_X(\tau)$ is the one formed of a single impulse:

$$C_X(\tau) = k\delta(\tau)$$
$$R_X(\tau) = k\delta(\tau) + m_x^2$$
$$\rho_X(\tau) = \delta(\tau) \quad \text{for any } \tau \text{ the variables are uncorrelated}$$
$$S_X(f) = k + m_X^2\delta(f)$$

Remember that:

$$\rho_X(\tau) = \frac{C_X(\tau)}{\sigma_X^2} = \frac{k\delta(\tau)}{k} = \delta(\tau)$$
$$\sigma_X^2 = C_X(0) = k\delta(0) = k$$

In nature white processes do **not** exist, otherwise they would have infinite power. They are the idealization of random processes with constant spectral power density in a big range W (order of THz).

$$S_X(f) = k\, rect(f/W) + m_X^2\delta(\tau)$$
$$R_X(\tau) \text{ is a very narrow cardinal sine}$$
$$R_X(\tau = 0) = P = kW + m_X^2$$
$$\sigma_X^2 = C_X(0) = R_X(0) - m_X^2 = kW$$
$$k = \sigma_X^2/W$$

8.4.1 Discrete white process

The idea is the same, but this processes **do** actually **exist** in nature:

$$C_X[m] = k\delta_m$$
$$R_X[m] = k\delta_m + m_X^2$$
$$\rho_X(\tau) = \delta_m \text{ for any } m \text{ all the samples are uncorrelated.}$$

$$S_X(\phi) = k + m_X^2 \delta(\phi)$$
$$R_X[0] = P = k + m_X^2$$
$$\sigma_X^2 = C_X[0] = R_X[0] - m_X^2 = k$$

But know that this processes actually exist in nature.

8.5 Continuous colored processes

If the process $y(t)$ is **colored**, its autocorrelation function is *wider* than an impulse and its power spectrum is shaped consequently.

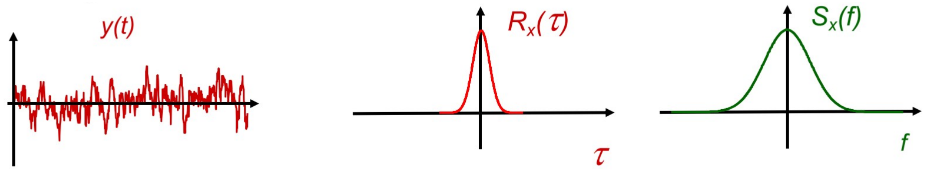


Figure 8.2: Continuous colored process.

8.6 White noise

A sequence of samples is said to be **white** if its samples are uncorrelated between them and consequently if the autocorrelation function is null for $\tau \neq 0$.

A sequence of samples is **colored** if its samples are correlated between them and the autocorrelation function is non-zero for $\tau \neq 0$.

The noise whose samples are **uncorrelated** is called **white noise**. The autocorrelation is therefore a Dirac delta centered in the origin.

8.6.1 AWGN

If the white noise is also **Gaussian** (it has Gaussian probability density), its samples are **statistically independent**.

The classical type of noise added in communication systems is called *Added White Gaussian Noise* (AWGN).

8.6.2 Coloring white noise

It is possible to *color* a white noise x_n filtering it with a filter that has response to the impulse h_n, obtaining the signal y_n.

Obviously the sampled signal $y(nT)$ is obtained convolving $x(nT)$ and $h(nT)$.

$$y_n = \sum_{k=0}^{K} x_{n-k} h_k = x_n * h_n$$

8.7 Transmission of a continuous random process through a LTI system

The system is deterministic, with input $x(t)$, output $y(t)$ and impulse response $h(t)$.

It can be shown that the mean value of the random process $y(t)$ in the input of a LTI system is given by the mean value of the input random process $x(t)$ times the transfer function $H(f)$ evaluated in 0:

$$\boxed{\mu_Y(t) = \mu_X(t) \cdot H(0)}$$

It can be also shown that the autocorrelation and the power spectrum of the random process $y(t)$ in the input of a LTI system can be obtained from:

$$\boxed{\begin{aligned} &R_y(\tau) = R_x(\tau) * h(\tau) * h^*(-\tau) \\ &S_Y(f) = |H(f)|^2 S_X(f) \end{aligned}}$$

8.8 Transmission of a discrete random process through a LTI system

The system is deterministic, with input x_m, output y_m and impulse response h_m.

$$\begin{aligned} &E[y_m] = E[x_m]H(0) \\ &R_y[m] = R_x[m] * h_m * h^*_{-m} \\ &S_Y(\phi) = |H(\phi)|^2 S_X(\phi) \end{aligned}$$

8.9 Cross-correlation between the output signal and the discrete input signal of a LTI system

$$\boxed{R_{yx}[m] = E[Y_{n+m}X_n] = R_x[m] * h_m}$$

In the case of a white input process:

$$R_{yx}[m] = \sigma_X^2 h_m + m_X^2 H(0)$$

The cross-correlation is equal to the impulsive response, scaled. It helps us to *estimate the transmission channel*: a *white* sequence is sent, with impulsive autocorrelation. Then, the autocorrelation with the output signal is computed, and this way the response to the impulse of the signal is computed.

8.10 Summary: stationary random processes

- Mean value of the process[1]:

$$m_x = E[x] = \int_{-\infty}^{\infty} x f_x(x)dx \quad \text{independent of } t$$

- Quadratic mean value and variance:

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x)dx = E[x(t)x(t)] \qquad \text{independent of } t$$

$$\sigma_x^2 = E[x^2] - m_x^2 \qquad \text{independent of } t$$

- Autocorrelation function:

$$R_x(\tau) = E[x(t+\tau)x(t)]$$

$$R_x(0) = E[x^2]$$

- Autocovariance function:

$$C_x(\tau) = R_x(\tau) - m_x^2$$

$$C_x(0) = \sigma_x^2$$

- Correlation coefficient:

$$\rho_x(\tau) = \frac{C_x(\tau)}{\sigma_x^2}$$

  If the autocovariance function is null, so is the correlation coefficient: $C_x(\tau) = 0 \Rightarrow \rho_x(\tau) = 0$. In this situation, we say that the samples are *uncorrelated*[2]. Only for Gaussian processes, *uncorrelation* and *statistical independence* are equivalent.
- Consider a LTI system. If we define the spectral power density as

$$S_x(f) = \mathcal{F}\{R_x(\tau)\}$$

  We have that:

$$R_y(\tau) = R_x(\tau) * h(\tau) * h^*(-\tau)$$

$$S_y(f) = |H(f)|^2 S_x(f)$$

- Power:

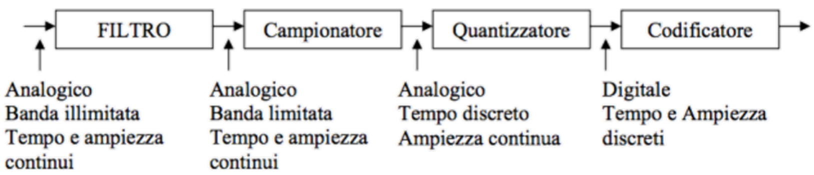
$$P = \int_{-\infty}^{\infty} S_x(f)df = R_x(0) = E[x^2]$$

[1]If the process is stationary, its statistic properties are independent of t.
[2]Keep in mind that this does not imply that samples are statistically independent.

9 | Quantizzazione di segnali discreti

9.1 Conversione analogico/digitale

The human representation of reality is **continuous** (analog world). However, the numerical *elaborators* manage **discrete** information (digital world). It is necessary to transform the analog signals into their digital equivalents.



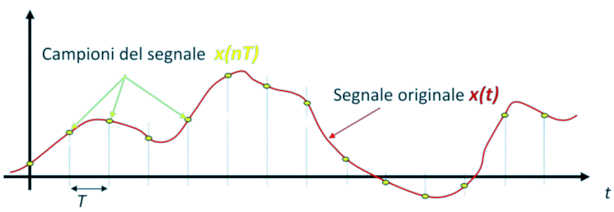
With the *sampling*, we limit the time instants on which the signal is defined. With *quantization* we do something similar but for the amplitudes. The signal will be able to take only certain values of the amplitude, not any in a known interval.

9.2 Numerical signals

A **numerical signal** $x_q(nT) = x_q[n]$ is a signal which is *discrete in time* and its samples have *discrete amplitudes*. In the case of the **discrete signal** $x(nT) = x[n]$, each sample is a real number that can take any value on a continuous interval.

If we wanted to represent each sample $x(nT)$ in a numerical way, it is necessary to **approximate** the real number with a finite number of K levels that cover the previously mentioned interval.

This operation is called **quantization**: associating to the amplitude of each sample the closest of the K *quantization levels* of the given interval.



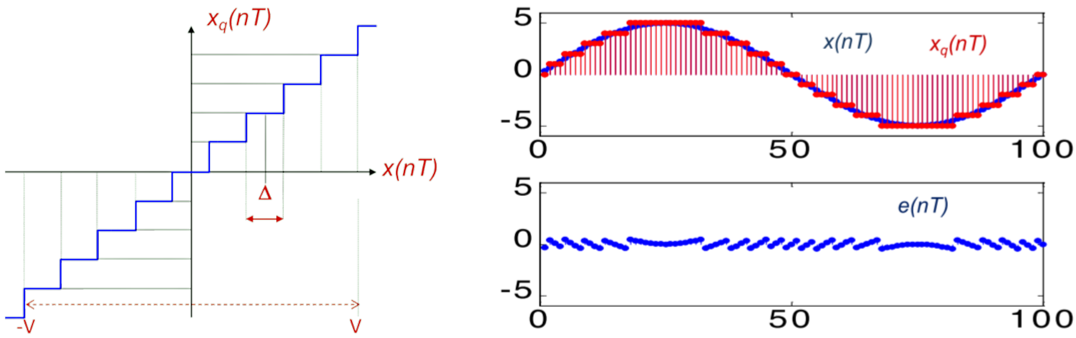
9.3 Quantization

The quantizator is a device which transforms the real sample $x(nT)$ into the sample $x_q(nT)$ with a number K of levels.

If the minimum and the maximum values that $x(nT)$ can take are $-V$ and V, the relation between the continuous value $x(nT)$ and the quantized $x_q(nT)$ is represented by a stair of K levels.

The quantization interval Δ is:

$$\boxed{\Delta = \frac{2V}{K}}$$



(a) Quantization levels. (b) Quantization error.

Figure 9.1: Quantization.

9.4 The quantization error

When quantizing the samples an error is made, which gets smaller as the number of levels K increases. The **quantization error** is defined as

$$\boxed{e(nT) = x_q(nT) - x(nT)}$$

If the number of levels K is high enough, the quantization errors of the samples become **uncorrelated** random variables, independent of $x(nT)$.

The discrete signal $x(nT)$ has a distribution of amplitudes more or less uniform between $-V$ and V, and the probability density of $e(nT)$ can be considered **uniform** between $-\Delta/2$ and $\Delta/2$.

Consequently the quantization error is a random variable with null mean value and variance equal to

$$\boxed{\sigma^2_{e(nT)} = \frac{\Delta^2}{12} = \left(\frac{2V}{K}\right)^2 \frac{1}{12} = \frac{V^2}{3K^2} = P_{e(nT)}}$$

Where $P_{e(nT)}$ is the power of the quantization error.

If we use N binary digits to represent the samples we will have that:

$$P_{e(nT)} = \sigma^2_{e(nT)} = \frac{\Delta^2}{12} = \frac{V^2}{3}\frac{1}{2^{2N}}$$

If we write the variance of the error in dBs, we obtain:

$$[\sigma^2_{e(nT)}]_{dB} = 10\log_{10}(\sigma^2_{e(nT)}) = 10\log_{10}\left(\frac{V^2}{3}\right) - 10\log_{10}(2^{2N})$$
$$= 10\log_{10}\left(\frac{V^2}{3}\right) - 10\log_{10}(4)\,N \approx 10\log_{10}\left(\frac{V^2}{3}\right) - 6N$$

Adding a binary digit (*bit*), the power of the quantization error is reduced by 6dB.

9.4.1 Natural encoding

Usually the quantization levels are associated to a number in binary format (binary coding).

With N binary digits (*bit*) $K = 2^N$ quantization levels are obtained. Therefore, a code of $N = \log_2 K$ bits can be associated to each level.

For example, if $N = 3$, we obtain $K = 8$ quantization levels V_m, which can be quantified in various ways with 3 bits.

Natural encoding: we associate binary numbers in increasing order to the quantization levels: 000, 001, 010, 011, 100, 101, 110, 111.

9.4.2 Gray's encoding

The natural encoding is not always the best choice to represent the levels of signals.

Gray's encoding: the binary numbers we associate differ by one bit between adjacent levels: 000, 001, 011, 010, 110, 111, 101, 100.

It is commonly used for the transmission of information as it reduces the probability of an error happening.

9.4.3 Signal to Noise Ratio

Supposing that the signal $x(nT)$ has a uniform distribution between $-V$ and $+V$, it is possible to determine the power as

$$P_{x(nT)} = \sigma^2_{x(nT)} = \int_{-V}^{V} a^2 \frac{1}{2V} da = \frac{V^2}{3}$$

The **Signal to Noise Ratio** (SNR) is the ratio between the power of the signal and the power of the quantization noise (error).

$$(SNR_q)_{dB} = \left(\frac{P_{x(nT)}}{P_{e(nT)}}\right)_{dB} = \left(\frac{V^2}{3}\frac{3}{V^2}2^{2N}\right)_{dB} \approx 6N$$

We have that:

$$SNR_q = \frac{P_{x(nT)}}{P_{e(nT)}} = 2^{2N} \qquad (SNR_q)_{dB} = \left(\frac{P_{x(nT)}}{P_{e(nT)}}\right)_{dB} \approx 6N$$

In practice, the necessary number of bits to quantize an specific signal is determined considering the signal's own SNR_x, and imposing that

$$SNR_q \geq SNR_x$$

# quantization bits	8	10	16
SNR (linear)	65536	1048576	4294967296
SNR (dB)	48	60	96

9.5 Quantization of non-uniform signals

If the signal $x(nT)$ does **not** have an uniform distribution between $-V$ and V, it is **not** convenient to use an uniform quantizer.

A better choice wold be to use a quantizer with variable quantizing interval to in order to adapt better to the statistics of the signal.

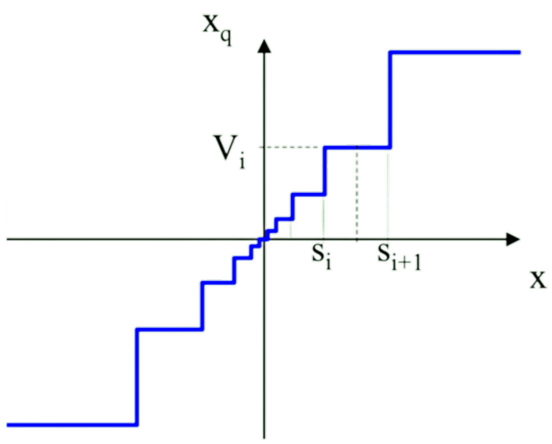
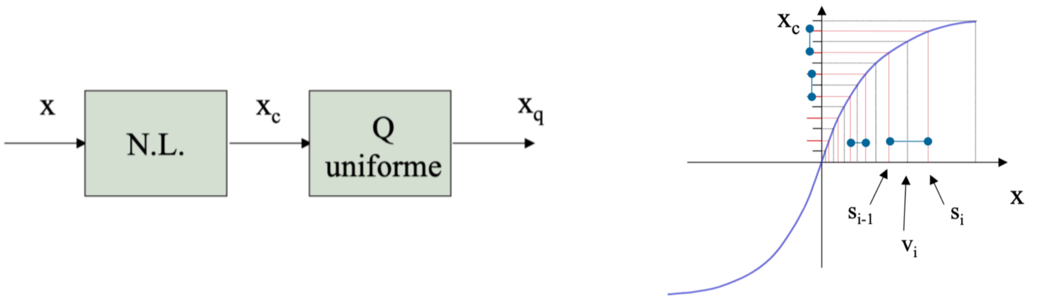


Figure 9.2: Non-uniform quantization.

To create a non-uniform quantizer a non-linear function is applied to the input (*distorter*), and then this signal is given to an uniform quantizer.

For instance, consider **logarithmic quantization**. The intervals closer to the origin are expanded, and the levels close to the maximum are compressed.

The process that has just been discussed is summarized in the figure 9.4.



(a) (b)

Figure 9.3: Non-uniform quantization.

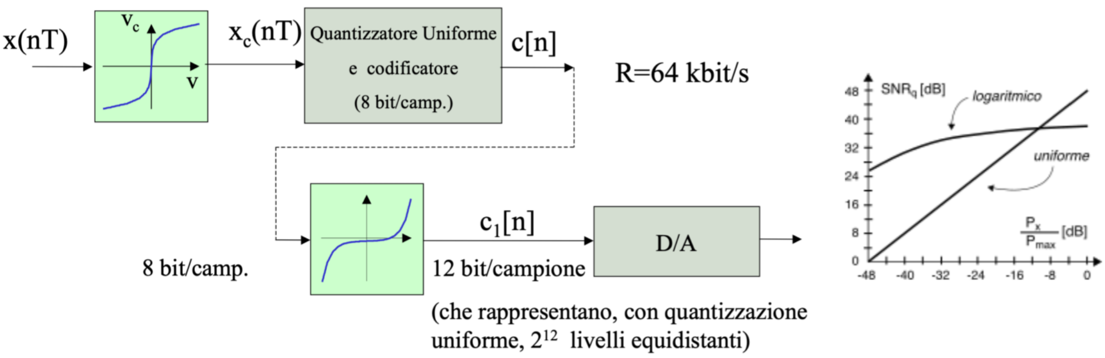


Figure 9.4

9.6 Bit-rate

After the binary coding, the numeric signal becomes a sequence of bits. The **cadence of bits per second** of a numeric signal is called **bit-rate**.

For a time-continuous signal sampled with frequency $f_c = 1/T$ and quantized using $N = \log_2 K$ bits for each sample, the bit-rate is:

$$R_b = \frac{\log_2 K}{T} = N\, f_c \; [bit/s]$$

When a signal is transformed into a sequence of bits, its origin is no longer important for its memorization/transmission.

The typology of the signal source (band, duration, amplitude) is a factor which only matters in the phase of going from analogical to digital and vice versa. The analysis and elaboration of numerical signals leaves aside the nature of the analogical source, given that the characteristics of the numerical signals are determined by the bit-rate.

9.6.1 Example: telephonic signal

The signal $x(t)$ is continuous in time and its maximum frequency is $3.6kHz$. The sampling theorem imposes a sampling frequency f_c bigger than $7.2kHz$. Normally $f_c = 8kHz$ is used (8000 samples per second).

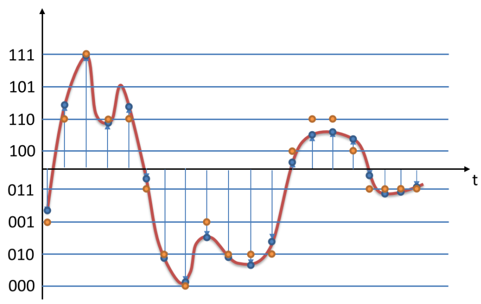
If we quantize the signal with $K = 256$ levels (symbols), $N = 8$ bits are enough. Consequently the numeric telephonic signal will have a bit-rate of:

$$Nf_c = 8 \cdot 8000 = 64\,\text{Kbit/sec}$$

Segnale	Banda	Frequenza di campionamento	Livelli di quantizzazione	**Flusso binario**
Segnale telefonico	300-3400 Hz	8000 Hz	256 livelli (8 bit)	**64 kb/s**
Voce	300-8000 Hz	16000 Hz	65536 livelli (16 bit)	**256 kb/s**
Musica	100-20 kHz	44.1 kHz	65536 livelli (16 bit)	**704 kb/s**
TV (PAL)	0 - 5 MHz	10 MHz	16.777.216 livelli (colori) (24 bit)	**240 Mb/s**
Cinema	0-500 MHz	1 GHz	16.777.216 livelli (colori) (24 bit)	**24 Gb/s**

Figure 9.5: Bit-rates of standard signals.

9.7 Summary: digitalization of a signal



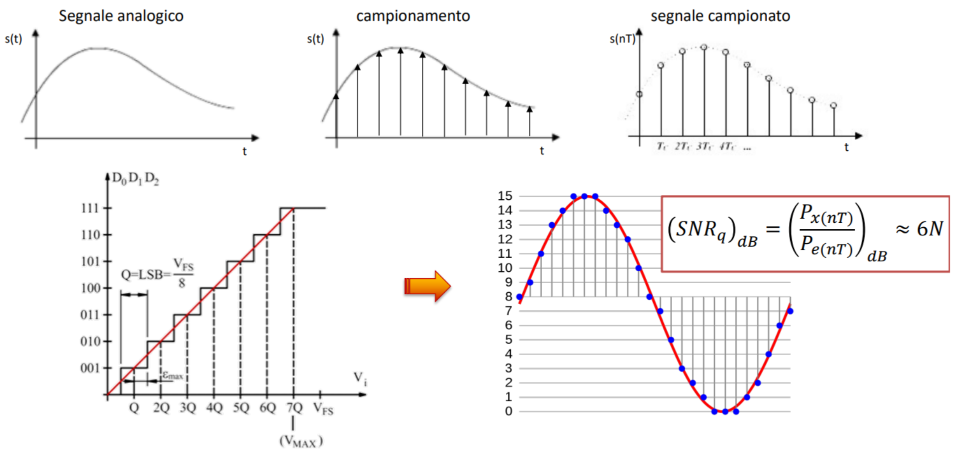
Sampling: measure of the amplitude of the signal in specific time instants which are equidistant. If $f_c > f_{\text{Nyquist}}$ the information the signal contains is completely preserved. Discrete amplitude levels are introduced.

Quantization: representation of the continuous amplitude of the signal, sampled using discrete levels. A specific group of bits is associated to each discrete level.

Now the analogical signal is in digital form.

10 | Codifica di Sorgente

Digitalizaton of a time continuous signal:



10.1 Binary encoding

On the previous chapter we saw the **natural** and **Gray's** encodings:

Natural:	000 001 010 011 100 101 110 111
Gray:	000 001 011 010 110 111 101 100

Is this the best way to label levels with a binary code? Yes, if the levels are **uncorrelated** and **equiprobable**. However, one of our objectives must be to reduce as much as possible the amount of bits needed, as this will improve memorization and transmission processes. This topic is related to the *source encoding*.

10.2 Huffman's encoding

Variable length binary source encoding, proposed by David Huffman in 1951. It is very simple to implement and its performance is close to theoretical optimal values.

General principles:

1) The most probable symbols are codified with the *shortest* code words (formed of the lowest bit amounts).
2) No code word must be preceded by another code word.

Example: Follow the steps in figure 10.1, from bottom to top.

Symbol	Probability
1	0.02
2	0.29
3	0.03
4	0.04
5	0.33
6	0.04
7	0.06
8	0.19

Symbol	Probability
5	0.33
2	0.29
8	0.19
7	0.06
4	0.04
6	0.04
3	0.03
1	0.02

Symbol	Probability
5	0.33
2	0.29
8	0.19
7	0.06
Node 1	0.05
4	0.04
6	0.04

1) Reorganize the table according to the probability of the levels, in **decreasing** order.

2) Select the two lowest probabilities (in this case 0.02 for 1 and 0.03 for 3) and generate a **sum node**: Node 1, 0.05.

3) The table needs to be updated now, adding the Node 1 and removing the two rows that they have generated it.

Symbol	Probability
5	0.33
2	0.29
8	0.19
Node 2	0.08
7	0.06
Node 1	0.05

Symbol	Probability
5	0.33
2	0.29
8	0.19
Node 3	0.11
Node 2	0.08

Symbol	Probability
5	0.33
2	0.29
8	0.19
Node 4	0.19

4) Once again, chose the two lowest probabilities (4 and 6, both with 0.04), sum them creating Node 2 (probability 0.08). Update the table.

5) Repeat steps 3 and 4 until getting the total probability of 1.

6) The tree graph will the be complete.

7) A bit **1** is assigned to each branch going to the **left**, and a **0** to each one going to the **right**.

8) Start reading from the top (node with probability of 1) down to each square, and obtain the code for each symbol.

Symbol	Probability
Node 5	0.38
5	0.33
2	0.29

Symbol	Probability
Node 6	0.62
Node 5	0.38

Symbol	Probability
Node 7	1

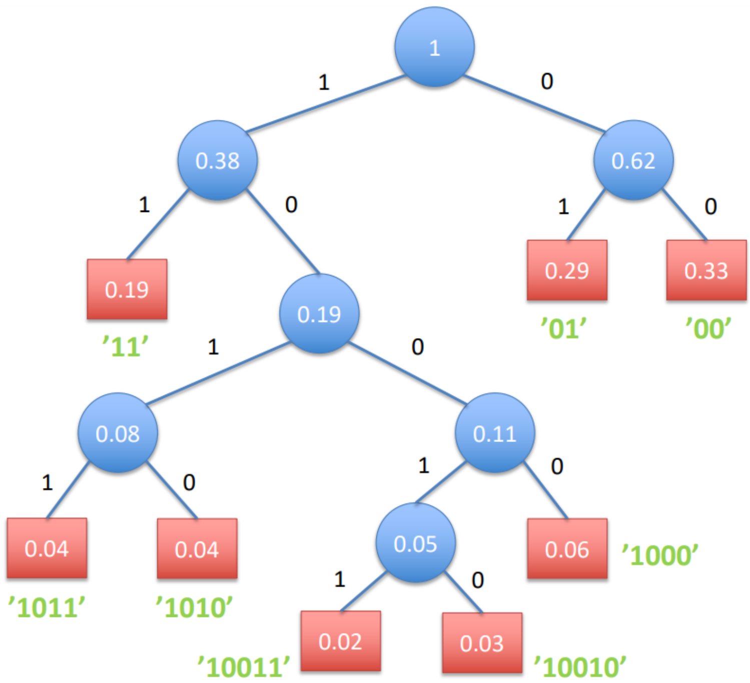


Figure 10.1: Huffman encoding: example.

Summarizing, this is what we get:

Symbol	Probability	Fixed encoding	Huffman encoding
1	0.02	000	10011
2	0.29	001	01
3	0.03	010	10010
4	0.04	011	1010
5	0.33	100	00
6	0.04	101	1011
7	0.06	110	1000
8	0.19	111	11

Average bit numbers per level:

$$\sum_{i=1}^{8} P_i K_i$$

Using the **binary encoding** with fixed length (3 bits/level):

$$\sum_{i=1}^{8} P_i K_i = 3 \sum_{i=1}^{8} P_i = 3$$

Huffman encoding:

$$\sum_{i=1}^{8} P_i K_i = 0.33 \cdot 2 + 0.29 \cdot 2 + 0.19 \cdot 2 + 0.06 \cdot 4 + 0.04 \cdot 4 + 0.04 \cdot 4 + 0.03 \cdot 5 + 0.02 \cdot 5 = 2.43$$

10.3 Entropy of the source

With Huffman's encoding we have reduced the needed bit amount to represent each level. In fact, there is a limit to the average number of bits needed to represent the samples of a numeric signal.

This value is called **entropy of the source**, and is indicated with H. the entropy measures the complexity of the signal, of the information amount that it contains, and consequently, the difficulty to transmit it.

The entropy represents the minimum average bit number per symbol **needed** to represent correctly an specific numeric signal.

Therefore, a numeric signal can be compressed **without information loss**, reducing the average bit number per symbol until the value of the entropy. A further reduction implies a information loss, the signal is not *loyal* to the original, and some parts will be missing.

10.3.1 Expression for the entropy

Shannon showed that *a random information source can not be represented with a number of bits smaller than its entropy.*

Hypothesis:

- Numeric signal, represented with M independent symbols.
- The i-th symbol appears with probability P_i.
- The number of symbols (duration of the signal) $N \to \infty$.

The information related to the i-th symbol is given by the $P_i N$ positions on which it appears, from a total of N available positions:

$$\binom{N}{P_i N} = \frac{N!}{(N - P_i N)!(P_i N)!} \approx (P_i)^{-NP_i}(1 - P_i)^{-N(1-P_i)}$$

where Stirling's formula $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$ has been used.

To be able to code the i-th level, we need to be able to code all the possible NP_i positions with respect to the N samples. To represent all the possible configurations we will need a number of bits equal to:

$$\log_2 \binom{N}{P_i N} = -NP \log_2 P_i - N(1 - P_i) \log_2(1 - P_i) \quad \text{[bit]}$$

Therefore, the average number of bits needed per symbol will be:

$$\frac{1}{N} \log_2 \binom{N}{P_i N} = -P_i \log_2 P_i - (1 - P_i) \log_2(1 - P_i) \quad [bit/symbol]$$

where $-P_i \log_2 P_i$ is the mean number of bits to codify the generic i-th symbol, and $-(1 - P_i) \log_2(1 - P_i)$ is the mean number of bits ti codify the rest of $M - 1$ symbols.

If all the symbols are **independent** between them, it can be shown that the minimum average number of bits needed to encode the source of M symbols is the number of bits needed to represent the configuration inside the set N, this is:

$$\boxed{H = \sum_{i=1}^{M} -P_i \log_2 P_i \quad \text{[bit/symbol]}}$$

If the symbols are equiprobable ($P_i = 1/M$), the maximum entropy is equal to

$$\boxed{H = \sum_{i=1}^{M} -P_i \log_2 P_i = \sum_{i=1}^{M} -\frac{1}{M} \log_2 \frac{1}{M} = \log_2 M \quad \text{[bit/symbol]}}$$

Example: going back to our example from section 10.2, we said that for the fixed length encoding and for the Huffman encoding the average bits per level were:

$$\sum_{i=1}^{8} P_i K_i = 3 \sum_{i=1}^{8} P_i = 3 \qquad \sum_{i=1}^{8} P_i K_i = 2.43$$

The **entropy of the source** is:

$$H = \sum_{i=1}^{8} -P_i \log_2 P_i \approx 2.38$$

Attention: Shannon's theorem says that there exists a encoding that allows to reach the number of bits of the entropy, but it does not specify what procedure to apply in each situation.

10.4 Sources with memory

The computation of entropy we have just considered is only valid for sources whose symbols are independent between them. If this is not true,

$$H < \sum_{i=1}^{M} -P_i \log_2 P_i$$

Sources with memory: sources in which different symbols are no longer independent but show a degree of **correlation** between them.

Correlation between symbols means that the value taken by a sample in a known time instant is predictable as a function of the samples from the past, and therefore the source shows some **redundancy**.

A good source encoding will delete as much of this redundancy as possible (thus the *memory*) to maintain uniquely the fundamental information, the non-predictable one.

Example: the method to encode sources with memory consists on combing between them more elementary symbols, creating higher level symbols which have smaller (or null) correlation degree.

Take for example a random variable x_n which can be take the value A or B:

$$\begin{cases} P(x_n = A) = P(A) = \frac{1}{2} \\ P(x_n = B) = P(B) = \frac{1}{2} \end{cases}$$

Consider the joint probabilities:

$$\begin{cases} P(A|A) = \frac{9}{10} \\ P(B|A) = \frac{1}{10} \\ P(A|B) = \frac{1}{10} \\ P(B|B) = \frac{9}{10} \end{cases}$$

If we encoded the single symbols of the given source, we find necessary to use 1 bit per symbol. However, it is evident that the symbols are tightly related between them.

Probabilities of the pairs:

$$\begin{cases} P(AA) = P(A|A)P(A) = \frac{9}{10}\frac{1}{2} = 0.45 \\ P(BA) = P(B|A)P(A) = \frac{1}{10}\frac{1}{2} = 0.05 \\ P(AB) = P(A|B)P(B) = \frac{1}{10}\frac{1}{2} = 0.05 \\ P(BB) = P(B|B)P(B) = \frac{9}{10}\frac{1}{2} = 0.45 \end{cases}$$

The new symbols are no longer equiprobable, so we can apply Huffman's encoding. In this case the average bit number per symbol we can obtain reduces to 0.825<1.

Joint probabilities of the triplets:

$$\begin{cases} P(AAA) = \frac{9}{10}\frac{9}{10}\frac{1}{2} = \frac{81}{200} \\ P(ABA) = \frac{1}{10}\frac{1}{10}\frac{1}{2} = \frac{1}{200} \\ P(AAB) = \frac{1}{10}\frac{9}{10}\frac{1}{2} = \frac{9}{200} \\ P(ABB) = \frac{1}{10}\frac{9}{10}\frac{1}{2} = \frac{9}{200} \\ P(BAA) = \frac{1}{10}\frac{9}{10}\frac{1}{2} = \frac{9}{200} \\ P(BBA) = \frac{1}{10}\frac{9}{10}\frac{1}{2} = \frac{9}{200} \\ P(BAB) = \frac{1}{10}\frac{1}{10}\frac{1}{2} = \frac{1}{200} \\ P(BBB) = \frac{9}{10}\frac{9}{10}\frac{1}{2} = \frac{81}{200} \end{cases}$$

The average number of bits we can obtain in this case is reduced to 0.68<1.

The number of symbols to encode increases exponentially with the number of the consecutively grouped samples. This implies a bigger complexity of the encoder and the decoder.

10.5 Lempel and Ziv's encoding method (LZ777)

It is a method that eliminates the correlation between symbols, considering simultaneously their occurrence rate.

It is the base of the **PKZIP** algorithm, commonly used to compress files without information losses in PCs.

Procedure:

1) The encoded sequence is composed of symbols preceded by a pointer, which specifies if the current symbol is preceded by a group of m symbols which already are in the encoded sequence that starts d symbols before the current one: $(d, m)A$.
2) Each time a symbol is read in the sequence we want to encode, it must be controlled if the symbol is already contained in the encoded sequence. If it exists, the next symbol is read and we check the existence of the couple of symbols. Do not insert anything in the encoded sequence until a non-existing string of symbols is found.
3) At this point, the non-present symbol is inserted preceded by the pointer (d, m) which specifies the number of symbols m to be preceded to be read at distance d.

Substituting groups of symbols with simple pointers of few bits allow to reduce considerably the dimension of the encoded data.

As the encoding advances, it will be easier to find groups of symbols already appearing in the encoded sequence.

The more variable is the sequence we want to encode (*high entropy*), the less compressible it will be.

10.6 Source encoding with losses

Very often it is convenient to encode sources with a bit/symbol number smaller than their entropy: *lossy* encoding. In this case it is no longer possible to keep unaltered the amount of information, which is **reduced** during the encoding phase.

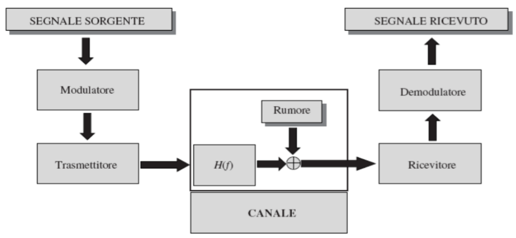
According to the type of data, the details that can be considered not very important are chosen to be lost. For example, in the `jpeg` encoding, this is done for images: the details at high frequency are reduced.

Figure 10.2: In these images the bit number is reduced by a factor of 2 each time.

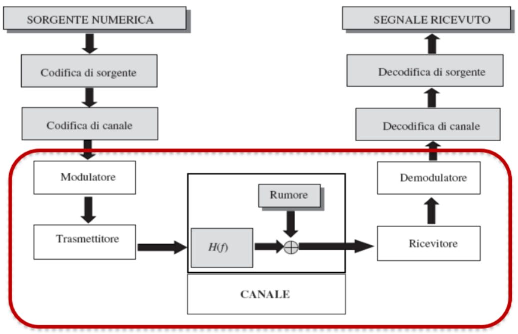
11 | Sistemi di trasmissione numerica

11.1 Sistema di comunicazione

The transmission through the physical channel is always done using analogical signals. The numeric source can come from an operation of sampling and quantization of a generic analogical source. The communication systems themselves can be for analogical or numerical signals.



(a) Communication systems for analogical signals.



(b) Communication systems for numerical signals.

Figure 11.1: Communication systems.

11.2 Transmission channel

The transmission medium is characterized by a **transfer function** in frequency $H(f)$ which determines the characteristics of the output signal given the input.

The principal alterations that the transmission medium can add are:

– Attenuation of the power of the signal as a function of its frequency and the distance traveled.

– Addition of a different delay for each frequency component of the signal (**dispersion**).

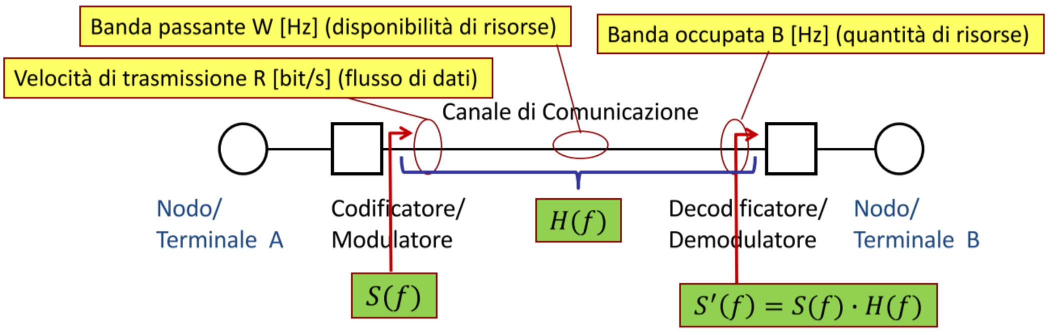
Each transmission medium has a region of the frequency domain on which the best response is found, in terms of attenuation and dispersion. This region is called **bandwidth of the channel**.

For a signal to be received as it has been transmitted, the bandwidth of the channel needs to be equal or bigger than the bandwidth of the signal itself. Otherwise, the signal loses some harmonics and is consequently distorted, this is, altered.

For any transmission medium the band with is reduced when the length of the medium is increased.

11.2.1 Occupied band vs bandwidth

The extension in frequency of the significant part[1] of the spectrum of the signal corresponds to the occupied band.

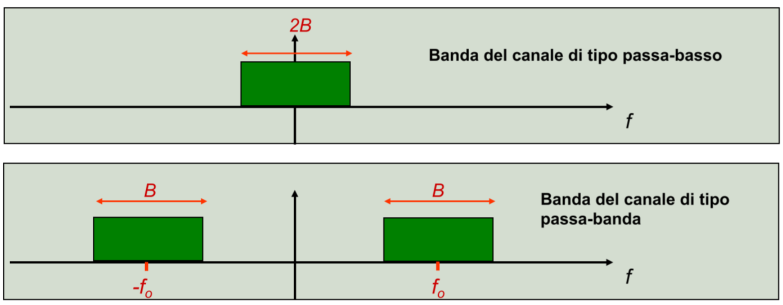


If the communication channel is **linear**, the spectrum of the received signal is equal to the spectrum of the transmitted signal times the transfer function of the channel.

To avoid the distortion the channel must modify the spectrum as less as possible.

$$\text{Bandwidth of the signal} > \text{Band occupied by the signal}$$

11.2.2 Channel types



Low-band transmission: the medium shows a bandwidth around $f = 0$.

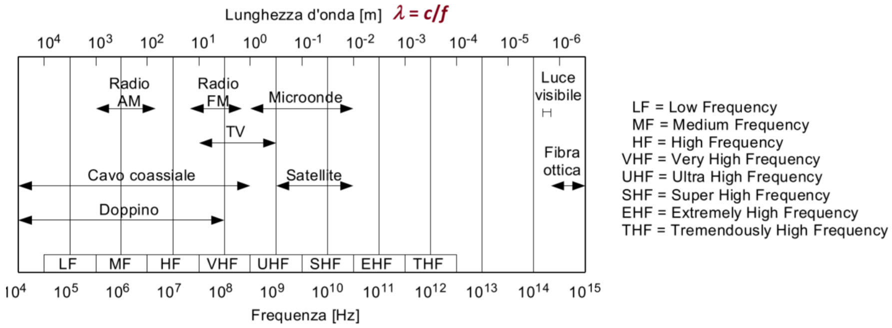
Band-pass transmission: the medium shows a bandwidth in different ranges of frequency. An EM wave (*carrier*) is used to transmit the signal.

The most used parts of the electromagnetic spectrum are the radio-waves and microwaves (*radio frequency*) and from infrared to visible light and ultraviolet for optical communications.

[1]For example the half amplitude band.

11.2.3 Band-pass transmission

In general, the higher the frequency of the carrier of a signal is, the bigger the available bandwidth for the transport of information, so **the bigger the amount of information we can transfer per time unit**.



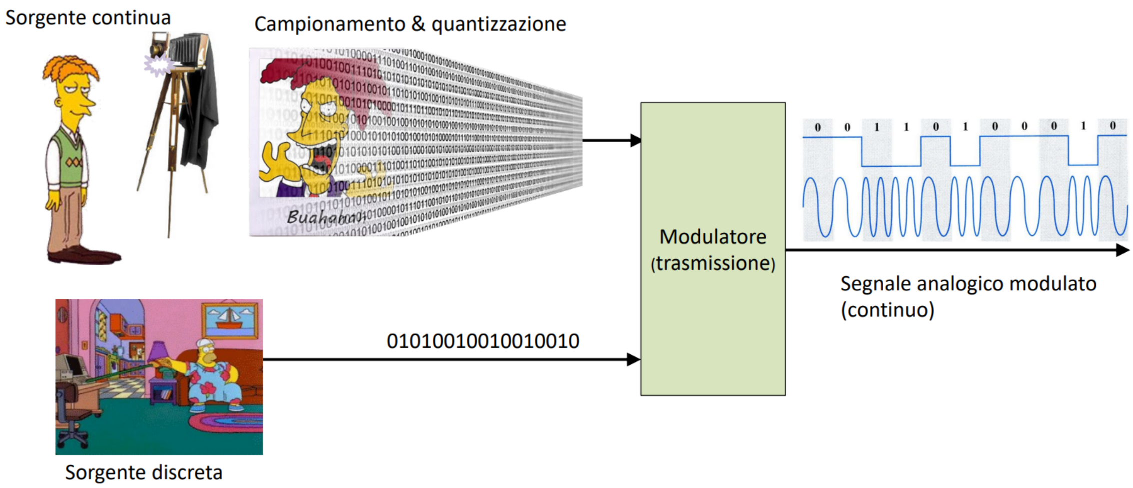
Note: Be careful with plots in logarithmic scale. The bandwidth for the optical fiber is much wider than the one of the coaxial wire.

11.3 Effect of the noise

It is well known that all the physical systems introduce a noise signal, that can be represented as a casual process that alter impulses during their generation, propagation and reception.

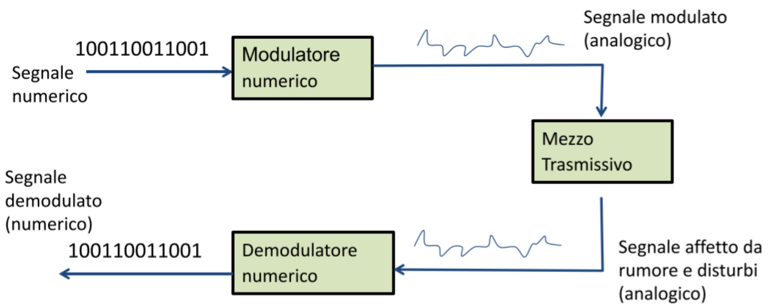
The typical effect of the noise consists on perturbing the form of the signal and adding errors in reception ⇒ Calculation of the error probability.

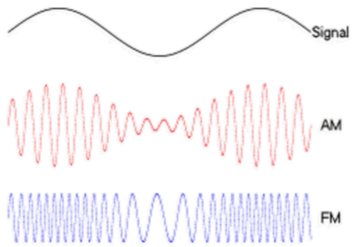
11.4 From the source to the information transmission



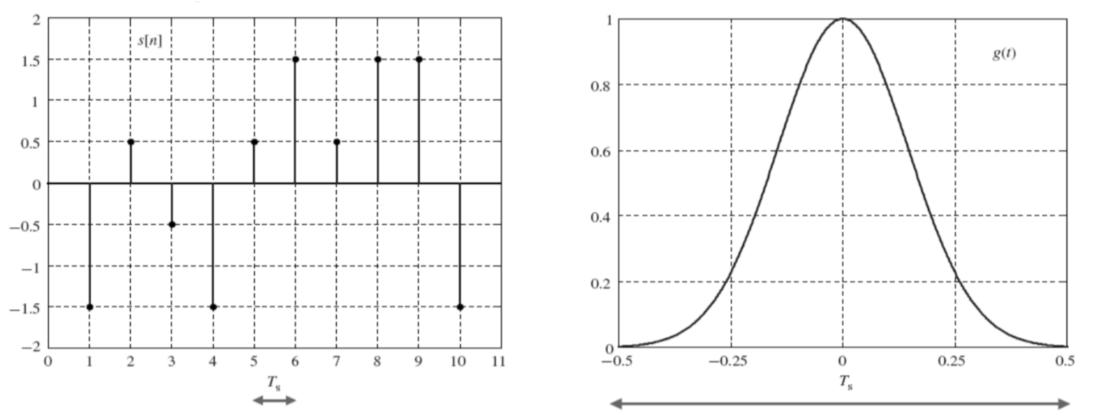
11.5 Modulation

The transmission of a digital (numeric) signal requires to create an opportune signal which is adapted to being transported in the transmission medium. The digital sequence is used to modify (*modulate*) some parameter of the signal (*modulated*) sent to the transmission medium.





11.5.1 Example: transmission of a 4 level numeric signal



(a) Numeric source $s[n]$.

(b) Analogical signal $g(t)$. Its f_{max} needs to be smaller than B and needs to have null value in correspondence of the times which are multiples of the time of the symbol (T_s). It is called **base band impulse**.

Figure 11.2: Example.

From the multiplication of the previous two signals we get $x(t) = \sum_n s_n g(t - nT_s)$.

If the transmission channel is ideal (it does not add any noise and does not alter the signal), sampling the received signal $x(t)$ with step T_s the sequence s_n can be obtained, and therefore, the transmitted bit sequence.

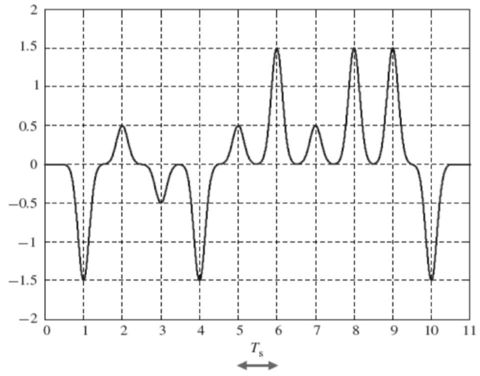
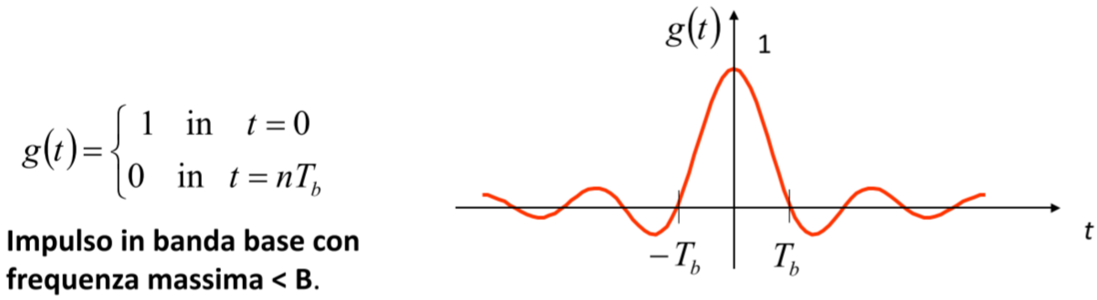


Figure 11.3: Example: $x(t) = \sum_n s_n g(t - nT_s)$.

11.6 Binary transmission in base band

$T_b = \frac{1}{\text{bit rate}}$ is the time that elapses between one bit and the next one, and it is called **bit time**.



The base band impulse is multiplied by a coefficient associated to the bit that we want to transmit. In the case of binary transmission there will be only two values of the multiplicative coefficient: c_1 and c_2.

Consequently, depending on the bit the following signals are transmitted: $c_1 g(t)$ or $c_2 g(t)$.

11.6.1 Pulse Amplitude Modulation (PAM)

The choice of the values of the coefficients is generally the **antipodal** one, which allows to reduce the transmitted power. Thus, $c_2 = -c_1$. See fig. 11.4.

$$...110... \quad \rightarrow \quad s_n = c_1\delta_{n+1} + c_1\delta_n - c_1\delta_{n-1}$$
$$x(t) = \sum_n s_n g(t - nT_b) = c_1 g(t + T_b) + c_1 g(t) - c_1 g(t - T_b)$$

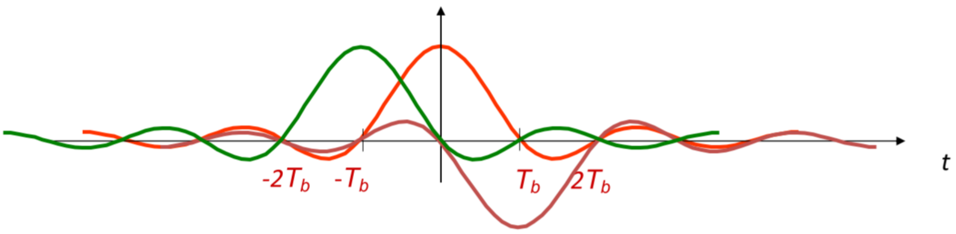


Figure 11.4: Pulse Amplitude Modulation: ...110...

11.6.2 Reception of a binary signal in base band

If the transmission channel is ideal (it does not add noise nor alter the sequence), sampling the received signal $x(t)$ with step T_b the sequence s_n is obtained, and thus, the sequence of the transmitted bits (fig. 11.5).

$$x(nT_b) = s_n = c_1\delta_{n+1} + c_1\delta_n - c_1\delta_{n-1}$$

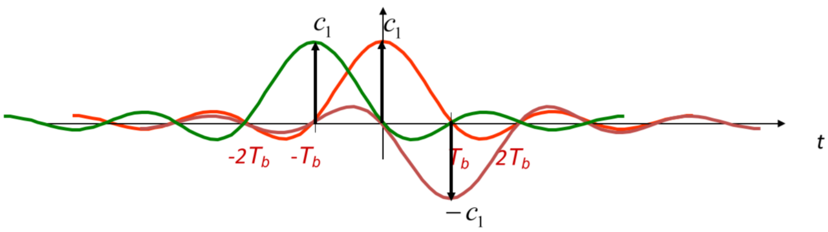


Figure 11.5: Pulse Amplitude Modulation. Received signal: ...110...

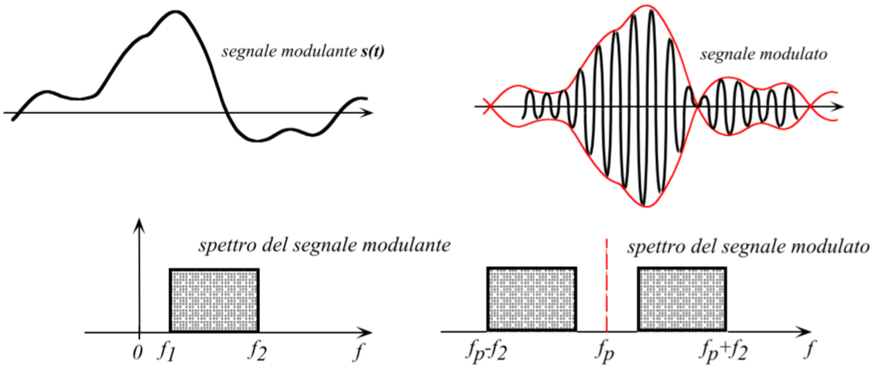
11.7 Transmission in pass band

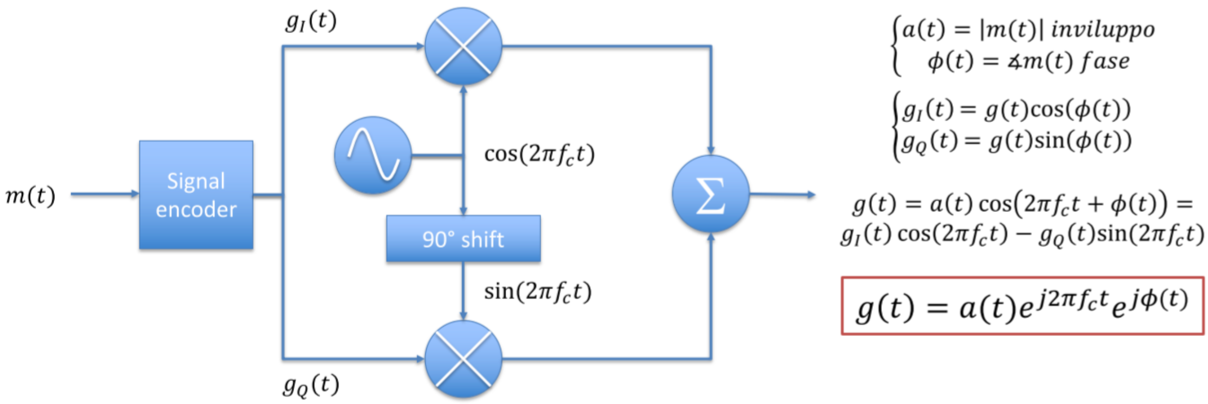
An electromagnetic wave[2] called **carrier** is used at a known frequency f_p to translate the spectrum of the signal towards the frequency of the carrier (Fourier transform's **modulation** property).

The band-pass signals are normally analyzed exploiting the representation of their **complex envelope** (*inviluppo complesso*). The signal $m(t)$ can also be complex: we can transmit the modulus in phase, and the phase in quadrature. Therefore, using this system we can simultaneously send two real signals.

Note that the signal after Σ must be real, as if it was not, we would not be able to transmit it. This means that its frequency spectrum has **hermitian symmetry**. The negative frequencies do not give additional information. This is taken into account to get the expression for $g(t)$.

[2]Mainly sines and cosines.

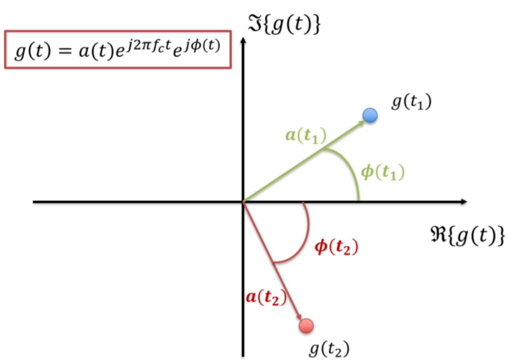




11.7.1 Geometrical representation of signals

The signals are represented as points of the complex plane. The modulus of the signal is the distance from the origin to the point. The instantaneous phase of the signal is analogous to the angle with the positive real axis.

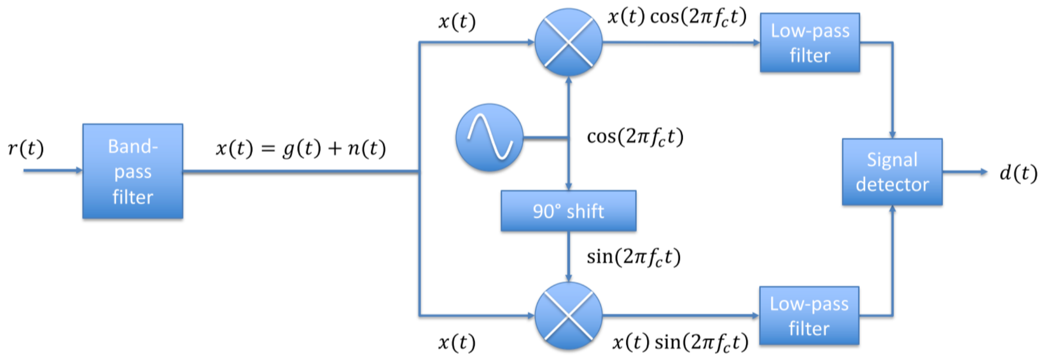
At each time t_1 it will be possible to associate a different point of the complex plane. The effect of the f_c frequency term is a rotation of the point along a constant-modulus circumference.



In most of the cases the signals are modulated in phase or/and amplitude, so in many cases we can ignore the term $e^{j2\pi f_c t}$, as we know exactly what it does.

11.7.2 Reception of band-pass signals

The received signal is filtered together to the noise added during the transmission, to limit the amount of noise as much as possible. Then the signals is **demodulated** on the two quadratures. It is important to use the same f_c used before.



$$x(t)\cos(2\pi f_c t) = g(t)\cos(2\pi f_c t) + n(t)\cos(2\pi f_c t)$$
$$= g_I(t)\cos(2\pi f_c t)^2 - g_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t) + n(t)\cos(2\pi f_c t)$$
$$= \mathbf{\frac{1}{2}g_I(t)} + \frac{1}{2}g_I(t)\cos(2\pi 2 f_c t) - \frac{1}{2}g_Q(t)\sin(2\pi 2 f_c t) + n(t)\cos(2\pi f_c t)$$

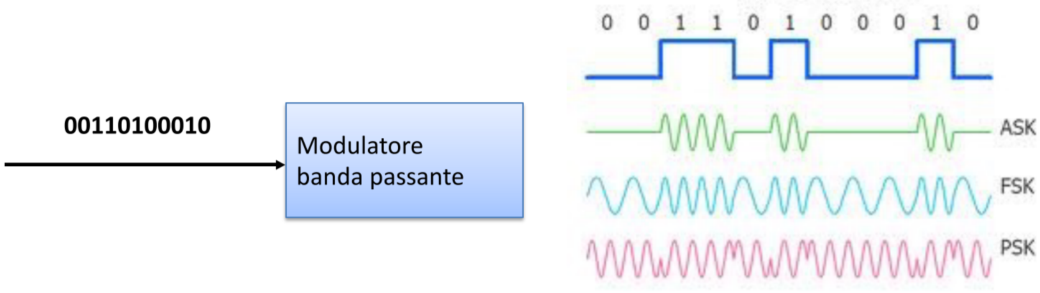
After the low-pass filter: $d_I(t) = \mathbf{\frac{1}{2}g_I(t)} + n_I(t)$.

$$x(t)\sin(2\pi f_c t) = g(t)\sin(2\pi f_c t) + n(t)\sin(2\pi f_c t)$$
$$= g_I(t)\sin(2\pi f_c t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)^2 + n(t)\sin(2\pi f_c t)$$
$$= -\mathbf{\frac{1}{2}g_Q(t)} + \frac{1}{2}g_Q(t)\cos(2\pi 2 f_c t) + \frac{1}{2}g_I(t)\sin(2\pi 2 f_c t) + n(t)\sin(2\pi f_c t)$$

After the low-pass filter: $d_Q(t) = -\mathbf{\frac{1}{2}g_Q(t)} + n_Q(t)$.

11.7.3 Typical band-pass modulation

- **Amplitude modulation** (ASK): variation of the instantaneous amplitude of the carrier.
- **Frequency modulation** (FSK): variation of the instantaneous frequency of the carrier.
- **Phase modulation** (PSK): variation of the instantaneous phase of the carrier.
- **QAM modulation**: simultaneous variation of the amplitude and the phase (ASK+PSK).



Amplitude modulation ASK

The 0 and 1 bits are represented using two different amplitude levels. The carrier is multiplied with the values 1 and 0 depending on the bit we want to transmit. It is the simplest modulation that can be generated, but its resistance against noise is smaller.

Modulation with variable envelope: strong distortion in presence of non-linearities. These non-linearities are added by the channel depending on the instantaneous power: when the signal has a maximum, the non-linearities are added at their maximum, and when there is no signal, there is no addition. These can not be eliminated afterwards.

This kind of modulation was very used in the past (for the radio, for example), but nowadays it is not so popular (even though it is used for optical transmissions, for short transmissions).

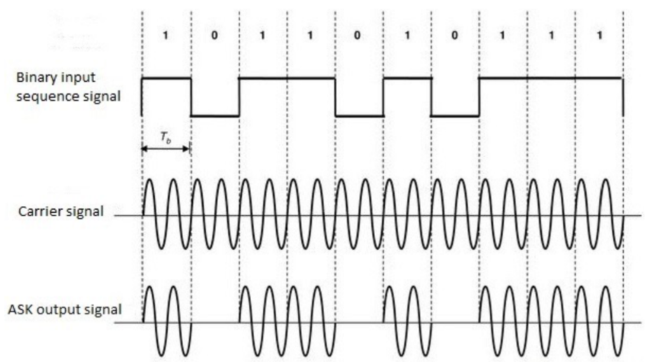


Figure 11.6: Amplitude modulation ASK.

Frequency modulation FSK

The 0 and 1 bits are represented using two different frequencies of the carrier. The signal is modulated at **constant envelope**: limited effect of the non-linearities added by the transmission medium.

The needed band is a bit wider than with ASK, as we are using two frequencies, but it is more resistant to noise.

Reception: frequency *discriminator*.

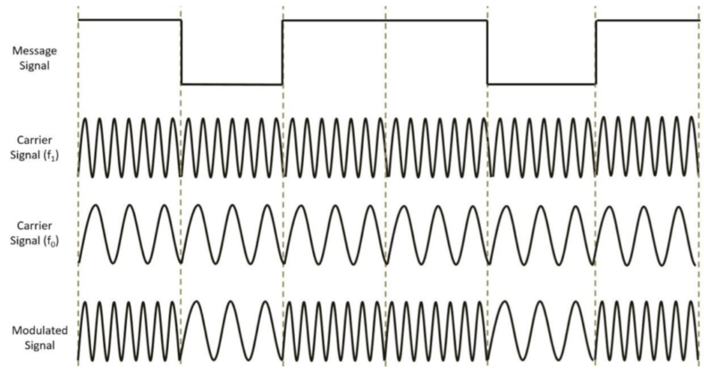


Figure 11.7: Frequency modulation FSK.

Phase modulation PSK

The bits 0 and 1 are represented using two different initial phases.

The signal is modulated at **constant envelope**: smaller impact of non-linearities of the transmission medium.

Reception: coherent demodulator or differential demodulator that acts on two consecutive symbols.

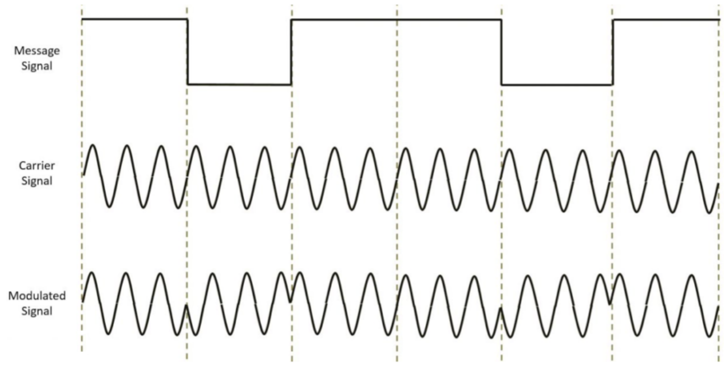


Figure 11.8: Phase modulation PSK.

Multilevel modulation

To increase the transmission capacity without modifying the occupied band, it is necessary to increase the **multilevel modulation**.

Example: PAM in base band or ASK in translated band.

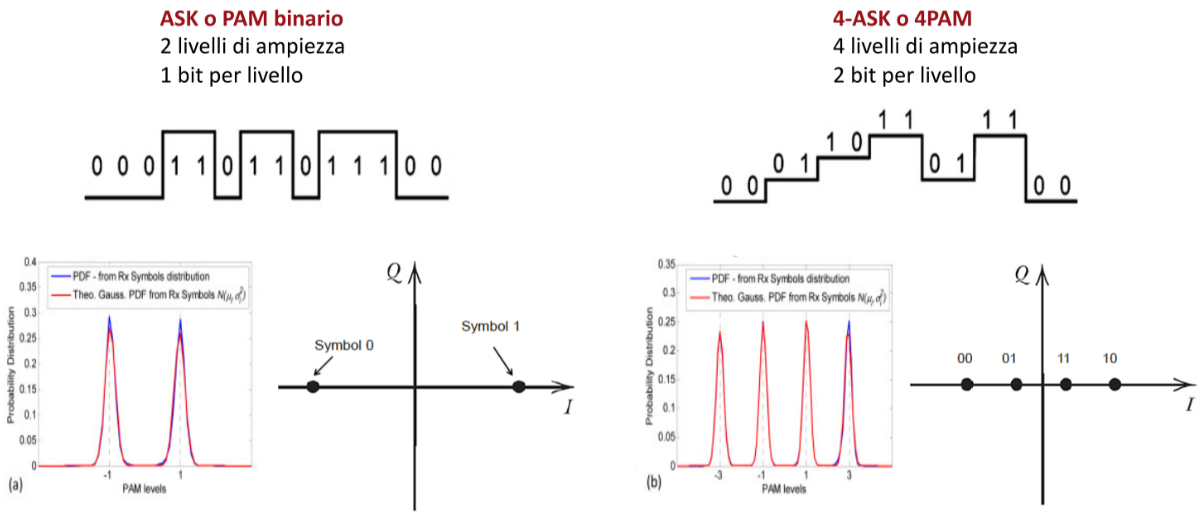
- The input bit flux is divided in groups of $\log_2 N$.
- N different *amplitude levels* are used.
- For each amplitude level transmitted (also called *symbol*), there are now logically corresponding $n = \log_2 N$ bits.

(a) Binary ASK or PAM: two amplitude levels ⇒ 1 bit per level.

(b) 4-ASK or 4PAM: 4 amplitude levels ⇒ 2 bit per level.

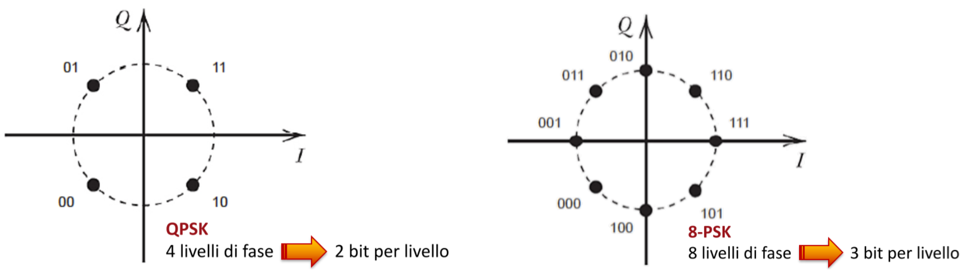
Figure 11.9: Multilevel modulation.

Multilevel amplitude modulation



Multilevel phase modulation

Example: PSK in translated band. The input bit flux is divided in groups of $\log_2 N$. N different *phase levels* are used. For each transmitted level (also called *symbol*), there are logically corresponding $n = \log_2 N$ bits.



(a) QPSK. (b) 8-PSK.

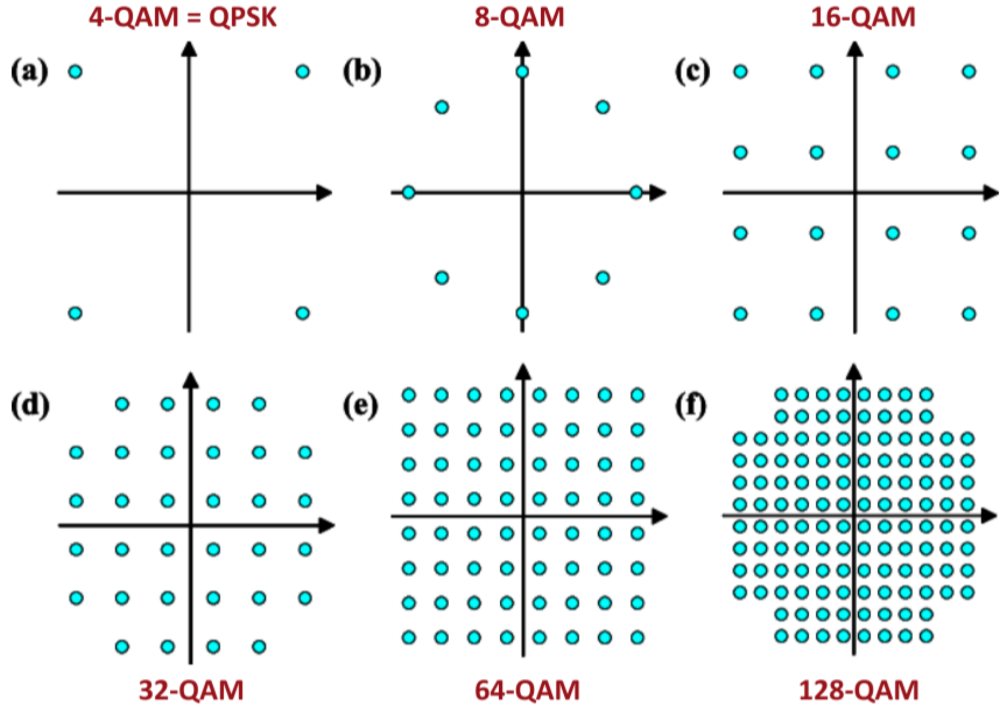
Figure 11.10: Multilevel phase modulation.

Notice that the labels of the different phase levels are assigned using Gray's encoding, which is more resistant to noise.

Amplitude and phase multilevel modulation

Be careful with the noise, we can not infinitely increase the levels, as with higher number of levels, less noise is needed to have problems.

Example: QAM in translated band. The bit flux is divided in groups of $\log_2 N$. We use N different amplitude and phase levels. For each transmitted level (also called *symbol*), there are logically corresponding $n = \log_2 N$ bits.



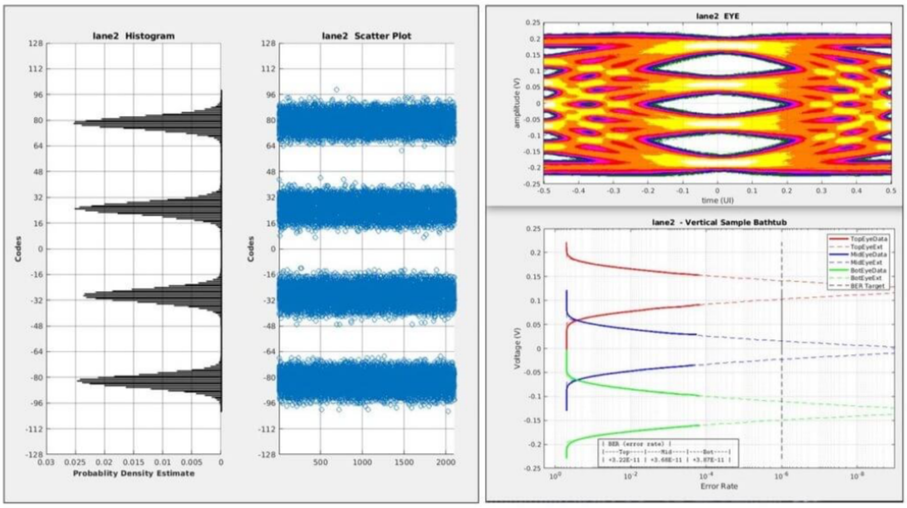
In radio, 1024QAM: single impulse 10 bits.

16QAM: optics, nowadays trying 64 bits (32 is not used because it is not square).

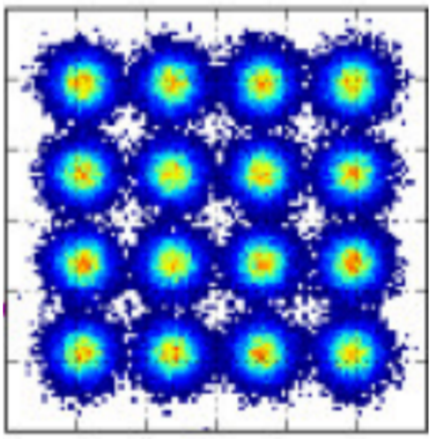
The **multilevel modulation** is used with **two objectives**:

- Increase the transmission velocity, keeping the same frequency band occupation.
  - $R \equiv$ permitted rate for a binary transmission with occupied band B.
  - $nR \equiv$ allowed rate for a multilevel transmission with occupied band B.
- Decrease the used resources, in terms of occupied band, keeping the bit-rate unaltered.
  - $B \equiv$ necessary band to make a binary transmission at bit-rate R.
  - $B/n \equiv$ necessary band to make a binary transmission at bit-rate R.

The number of levels (and therefore the number of bits carried by each level) cannot be increased arbitrarily due to noise which can introduce misunderstanding in reception (reception error). See calculation of the probability of error.



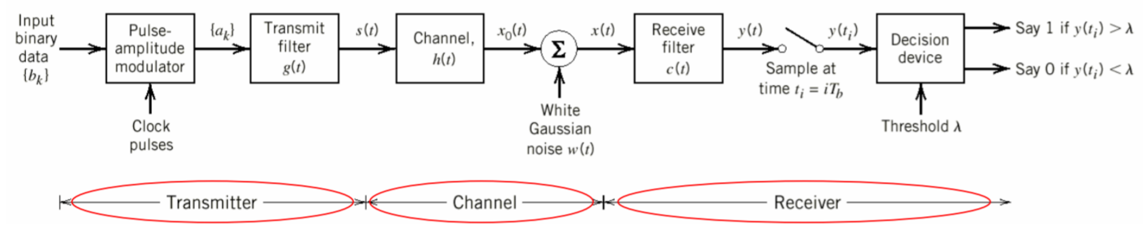
(a)



(b)

Figure 11.11: Multilevel modulation.

12 | ISI and matched filter



12.0.1 Transmitted signal (base band)

The shape of the single impulse is determined by $g(t)$, whereas the value taken at its maximum depends on the specific amplitude a_k.

In this way, it is possible to write analytically the signal, composed by all the impulses[1] consecutively:

$$s(t) = \sum_{k=-\infty}^{\infty} a_k g(t - kT_b)$$

In the following sections we will try to find the optimal $g(t)$.

12.0.2 Received signal (base band)

The signal $s(t)$ is propagated through the channel with transfer function $H(f)$. After the reception filter $c(t)$ we get $y(t)$. When sampling is done, it is possible to write the expression of the received signal this way:

$$y(t_i) = y(iT_b) = \sum_{k=-\infty}^{\infty} a_k p(iT_b - kT_b) + n(iT_b)$$

The function $p(iT_b - kT_b)$ represents the form of the base impulse $g(t)$, modified by the propagation through the medium. Distorted by $H(f)$ and $C(f)$. Knowing that:

$$s(t) = \sum_{k=-\infty}^{\infty} a_k g(t - kT_b) \qquad p(t) = g(t) * h(t) * c(t)$$

We have that:

$$y(t_i) = y(iT_b) = \sum_{k=-\infty}^{\infty} a_k p(iT_b - kT_b) + n(iT_b)$$

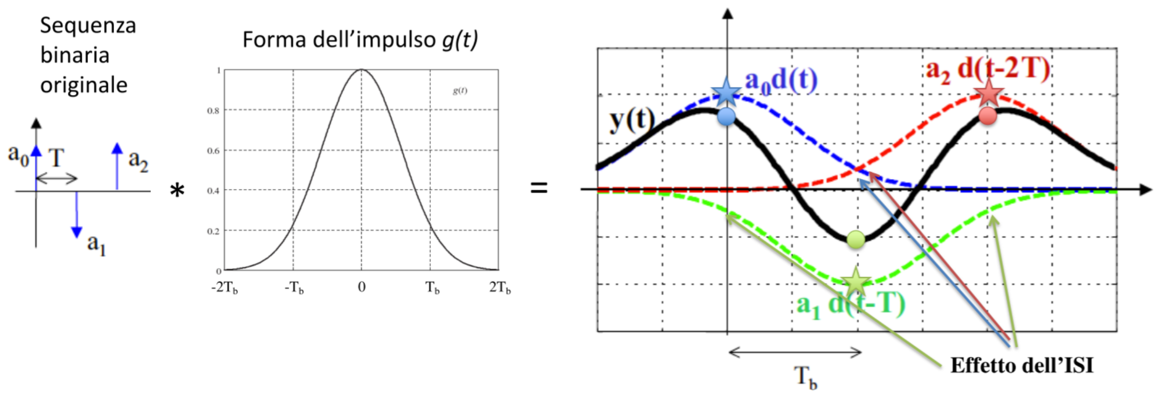
We can rewrite it this way:

$$y(t_i) = y(iT_b) = a_i + \sum_{k \neq i} a_k p(iT_b - kT_b) + n(iT_b)$$

[1]They are no longer Dirac's deltas, but copies of $g(t)$ that carry information.

where a_i is the value generated at time iT_b, $\sum_{k\neq i} a_k p(iT_b - kT_b)$ is the sum of the residue of the other k impulses present at time iT_b (with $i \neq k$) and $n(iT_b)$ is the value of the noise signal at the instant iT_b.

12.1 Intersymbolic interference



The dotted lines in the image in the right are the independent impulses. Their peak values are the stars, the ones we want to read in the received signal. However, see that the values we get are the sums, the ones given by the points in the thick continuous line.

The difference between stars and points is the **intersymbolic interference** (ISI). ISI signal:

$$\sum_{k\neq i} a_k p(iT_b - kT_b)$$

If this term is not null, the presence of other symbols adds another *noise* term to the signal, which can generate errors in the decision phase.

The condition to have **null ISI** is:

$$\sum_{k\neq i} a_k p(iT_b - kT_b) = 0$$

Therefore, the shape of the impulse has to be null at all the integer multiples of T_b, so that there is no effect of the ISI in the optimal sampling intervals. Optimal form:[2]

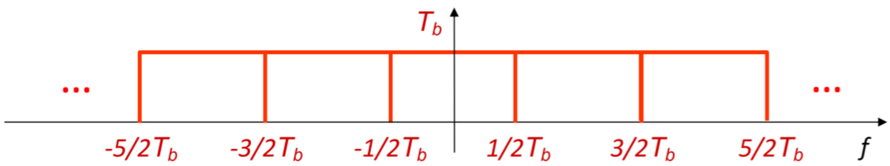
$$\boxed{p(t) = \frac{\sin \pi \frac{t}{T_b}}{\pi \frac{t}{T_b}}}$$

[2]It is not the only family of functions that can be used for this, but yes the one with the narrowest frequency band. That is the reason why we call it *optimal*.

If we write the condition to have *null ISI in the frequency domain*, we get **Nyquist's criteria for the ISI**:

$$\boxed{\sum_{k=-\infty}^{\infty} P\left(f - \frac{k}{T_b}\right) = T_b}$$

Explanation: we are sampling in time, so we are creating replicas in the frequency domain. To obtain a single impulse in time, we need to get a constant in frequency after creating all the replicas.



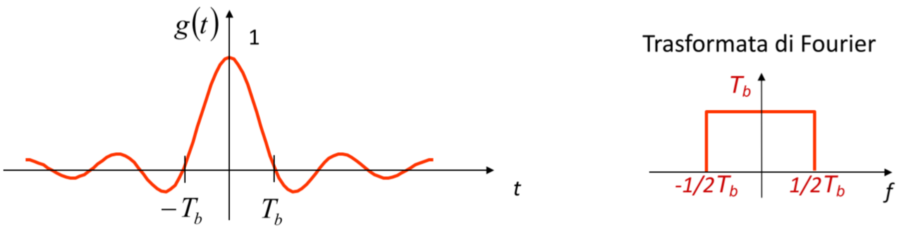
This criteria ensures that all the samples observed in the interval nT_b will be null, except from the one in the center (a constant value is obtained in frequency only if the signal in time is formed by a single impulse).

The function that occupies the least band in frequency and satisfies this relationship is the **rectangle**.

12.1.1 Ideal impulse's shape: cardinal sine

If $H(f)$ and $C(f)$ do not add any distortion to the signal,

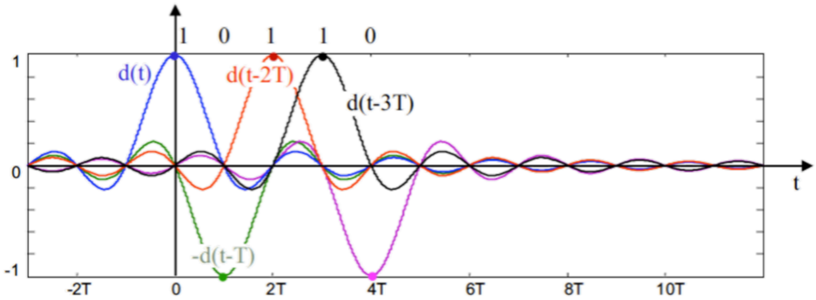
$$p(t) = g(t) = \frac{\sin \pi \frac{t}{T_b}}{\pi \frac{t}{T_b}} \quad \Rightarrow \quad G(f) = T_b \, rect(T_b f)$$



If the low pass channel has (unilateral) band B, the shortest **bit time** T_b that can be used is equal to $1/2B$, and thus the **maximum bit-rate** is equal to the **double of the band** ($2B$):

$$\frac{1}{2T_b} = B \quad \Rightarrow \quad \text{bit rate} = \frac{1}{T_b} = 2B$$

As we have said the cardinal sine is the best impulse form as it minimizes the band occupation, but it is not easy to use in reality. We take it as a **theoretical limit**.

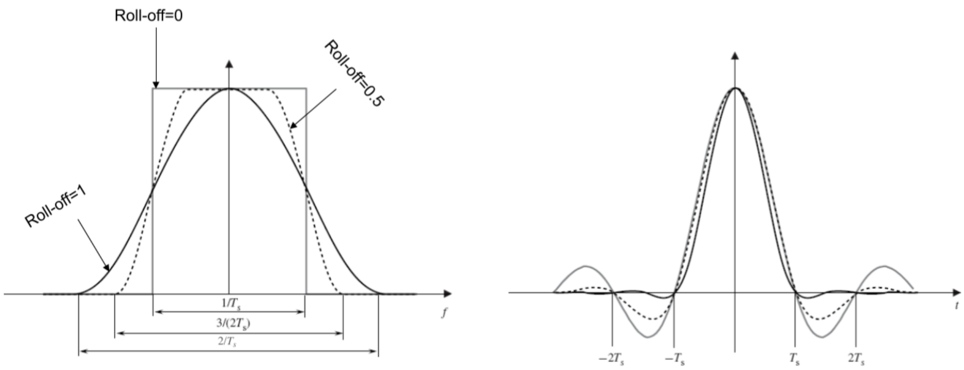


It is not physically doable, as it goes from $-\infty$ to ∞ and consequently has instantaneous transitions in frequency. A perfect synchronization would be needed, given that a small delay with respect to the optimal sampling instant would provoke a strong ISI to arise.

12.1.2 Raised-cosine impulses

In practice waveforms with wider band are used, with more gradual transitions and odd symmetry, with respect to the extreme frequencies of the cardinal sine, so as to respect Nyquist's criteria.

The transition at odd symmetry with respect to the frequency $\pm\frac{1}{2T_b}$ guarantees the annulment of the base band impulse at the time instant $t = nT_b$.



12.1.3 Expression of the raised-cosine

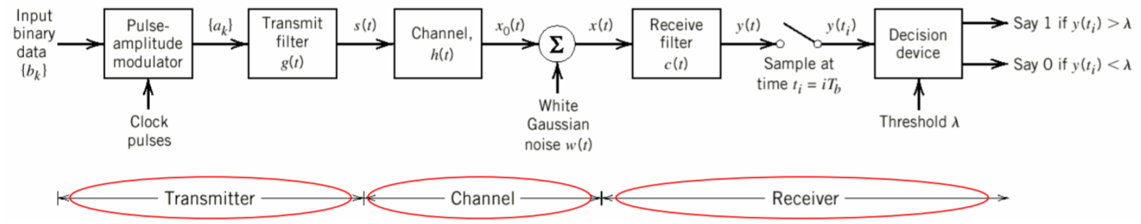
$$P(f) = G(f) = \begin{cases} T_b, & |f| \leq \frac{1-\alpha}{2T_b} \\ \frac{T_b}{2}\left[1 + \cos\left(\frac{\pi T_b}{\alpha}\left[|f| - \frac{1-\alpha}{2T_b}\right]\right)\right], & \frac{1-\alpha}{2T_b} \leq |f| \leq \frac{1+\alpha}{2T_b} \\ 0, & \text{otherwise} \end{cases}$$

The α roll-off factor indicates the excess band in comparison with the ideal case of the rectangle. The band and the bit-rate of a raised-cosine signal will therefore be:

$$B = \frac{1+\alpha}{T_b} \qquad R_b = (2-\alpha) \cdot B$$

with $B \leq R_b \leq 2B$.

12.2 Optimal receptor (base band)



Optimal reception of a single impulse in presence of noise.

- The receptor knows the form of the impulse $g(t)$.
- The noise is of type AWGN with null mean value and variance $\sigma_n^2 = \frac{N_0}{2}$.

$$y(t) = g(t) * c(t) + w(t) * c(t) = g_0(t) + n(t)$$

Optimal situation: maximize the energy of the signal, minimize the energy of the noise. This is equivalent to finding the $c(t)$ that allows to maximize the SNR: **matched filter**.

12.2.1 The matched filter

The optimal reception is obtained when we use the **matched filer** as reception filter. We call it *matched* because it matches the signal $g(t)$.

It allows to maximize the power of the signal and to minimize the power of the noise, this is, to minimize the SNR. Peak SNR (we are checking it in the maximum point of our signal):

$$\eta = \frac{|g_0(T)|^2}{E\{n^2(t)\}}$$

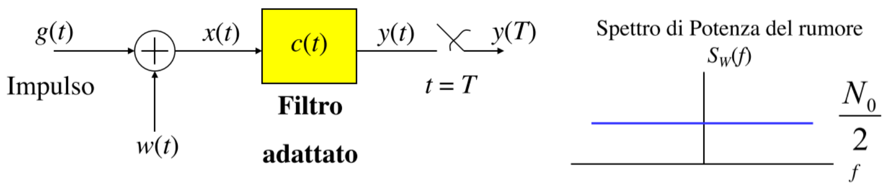
where the numerator is the instantaneous power at time T and the denominator is the mean power of the noise.

Mean power of the noise

$$n(t) = w(t) * c(t)$$
$$S_N(f) = S_W(f)S_C(f) = \frac{N_0}{2}|C(f)|^2$$
$$E\{n^2(t)\} = \int_{-\infty}^{\infty} S_N(f)df = \frac{N_0}{2}\int_{-\infty}^{\infty}|C(f)|^2 df$$

Instantaneous power of the signal

$$g_0(t) = g(t) * c(t) \rightarrow G_0(f) = C(f)G(f)$$
$$g_0(t) = \int_{-\infty}^{\infty} C(f)G(f)e^{j2\pi ft}df$$
$$|g_0(T)|^2 = \left|\int_{-\infty}^{\infty} C(f)G(f)e^{j2\pi fT}df\right|^2$$



12.2.2 Maximum value of the peak SNR

$$\eta = \frac{|g_0(T)|^2}{E\left\{n^2(t)\right\}} = \frac{\left|\int_{-\infty}^{+\infty} C(f)G(f)e^{j2\pi fT}df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{+\infty}|C(f)|^2df}$$

The matched filter allows us to maximize this relationship. Using the Schwartz inequality:

$$\left|\int_{-\infty}^{\infty} \phi_1(x)\phi_2(x)dx\right|^2 \leq \int_{-\infty}^{\infty} |\phi_1(x)|^2\, dx \int_{-\infty}^{\infty} |\phi_2(x)|^2\, dx$$

with $\phi_1(f) = C(f)$ and $\phi_2(f) = G(f)e^{j2\pi fT}$, we have:

$$\left|\int_{-\infty}^{+\infty} C(f)G(f)e^{j2\pi fT}df\right|^2 \leq \int_{-\infty}^{+\infty} |C(f)|^2df \int_{-\infty}^{+\infty} |G(f)|^2df$$

$$\eta = \frac{\left|\int_{-\infty}^{+\infty} C(f)G(f)e^{j2\pi fT}df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{+\infty}|C(f)|^2df} \leq \frac{\int_{-\infty}^{+\infty}|C(f)|^2df\int_{-\infty}^{+\infty}|G(f)|^2df}{\frac{N_0}{2}\int_{-\infty}^{+\infty}|C(f)|^2df} = \frac{2}{N_0}\int_{-\infty}^{+\infty}|G(f)|^2df = \eta_{\max}$$

This value represents the maximum of η. It does no depend on the expression of the frequency response of the filter $C(f)$. It only depends on the energy of the signal and the spectral power density of noise.

12.2.3 Expression of the matched filter

It is possible to find the expression of $C(f)$ that allows to obtain:

$$\eta = \frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2df = \eta_{max}$$

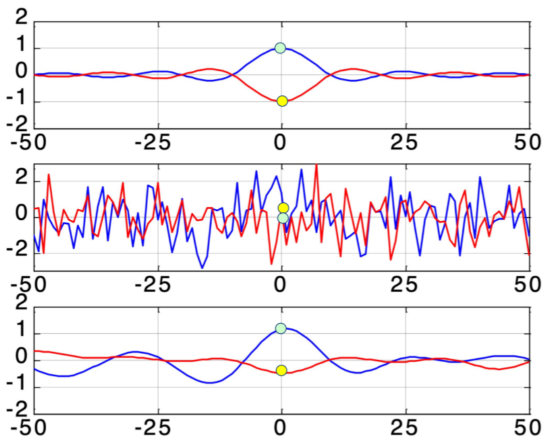
This is satisfied when:

$$C_{opt}(f) = kG^*(f)e^{-j2\pi fT}$$

The optimal filter is therefore linked to the Fourier transform of the impulse. Considering the signal is real, we obtain

$$G(-f) = G^*(f)$$
$$c_{opt}(t) = kg^*(T-t) = kg(T-t)$$

12.2.4 How the matched filter acts



Take two signals without noise $c_1g(t)$ and $c_2g(t)$. They are plotted in the first graph.

The second graph shows $c_1g(t)$ and $c_2g(t)$ with the added noise. The values at time $t = 0$ are almost equal.

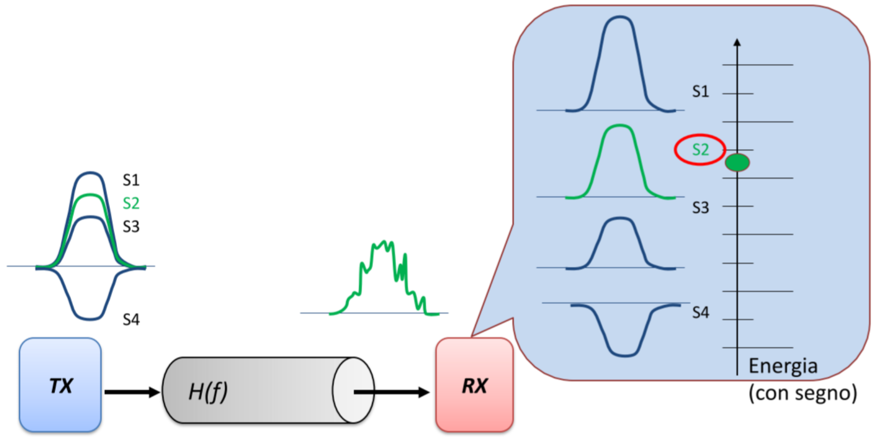
In the third graph we can see **the effect of the matched filter**. The values at $t = 0$ are again quite different.

12.3 Reception by correlation measurement

The response to the impulse of the matched filter is a time scaled, delayed and inverted version of the impulse used in transmission.

The reception of the information is based on a **measure of similarity** (**correlation**) between the received signal and the *sample* impulse we were expecting to receive.

In practice, the **matched filter** performs a time integral of the received impulse on an interval equal to the symbol duration, to which the integral of the power or the noise (generally a constant) through the band of the signal is summed.



12.3.1 Decision operation

After the matched filter, the *decision maker* elaborates the sample of the received signal at each T. It is a random number (estimation of the **energy** of the received signal) that is compared with the possible results expected.

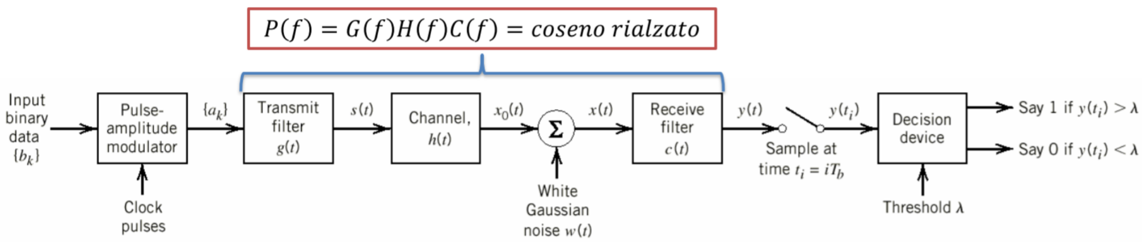
It is decided that the symbol whose expected level is closest to the measured received energy has been transmitted.

12.4 ISI + matched filter

To eliminate the intersymbolic interference, $P(f)$ needs to have a raised-cosine like expression.

Very frequently the **square root of a raised-cosine** is used as transmission impulse. In this way the adapted filter will have the same expression and the result of their multiplication will be a regular raised-cosine.

If it is known *a priori*, also $H(f)$ can be compensated in order to have a overall function equal to a raised-cosine.



13 | Probabilità di errore

13.1 Base-band transmission

We will start by analyzing the case of binary base-band transmission in presence of white Gaussian noise $w(t)$ and the matched filter.

13.1.1 Received signal

$w(t)$ is a Gaussian white noise with null mean value and bilateral spectral power density

$$S_w(f) = \frac{N_0}{2}$$

The power of the noise is $\sigma_{w_F}^2 = N_0 B$.

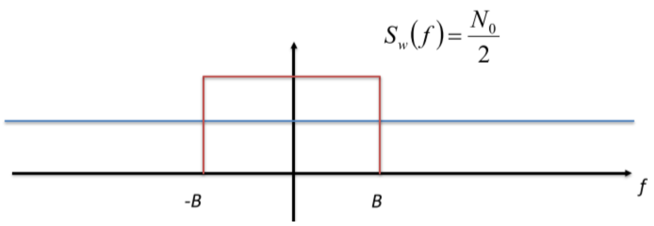


Figure 13.1: Remember that we said that in practice the impulses we used were not cardinal sines but raised-cosines. Considering this, the form of the filter in the frequency domain is not the rectangle shown.

We are considering this signal:

$$x(t) = \sum_k a_k g(t - kT_b)$$

With this,

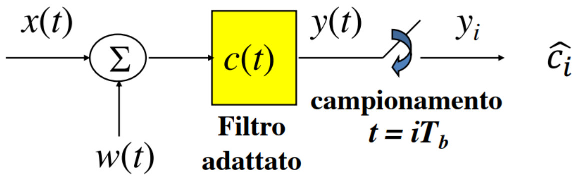
$$\begin{aligned} y(t) &= [x(t) + w(t)] * h(t) * c(t) \\ &= x(t) * h(t) * c(t) + w(t) * h(t) * c(t) \\ &= p(t) + w_F(t) \end{aligned}$$

13.1.2 Sampling at the receiver

Both the signal and the added noise are sampled:

$$y(iT_b) = x(iT_b) + w_F(iT_b) = a_i + w_F = \pm A + \frac{1}{T_b} \int_{iT_b}^{(i+1)T_b} w(t)dt$$

$$E = A^2 T_b \qquad k = \frac{1}{AT_b}$$



a_i can take the values $c_1 = -A$ or $c_2 = -c_1 = +A$. w_F is a Gaussian random variable with null mean value and variance $\sigma^2_{w_F}$.

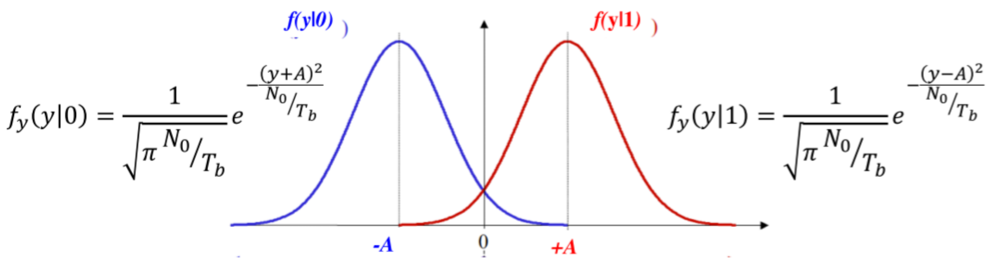
After making the low-pass filtering and the sampling, the following values are measured:

$$\widehat{c_1} = c_1 + w_F = -A + w_F \quad \text{if we have transmitted } c_1$$
$$\widehat{c_2} = c_2 + w_F = +A + w_F \quad \text{if we have transmitted } c_2$$

13.1.3 Distribution of the signal sampled on the receiver

The variable y_i is therefore a random process that can be represented as the sum of two Gaussian processes, given that it is composed by a determined value ($\pm A$) summed to the Gaussian process (noise) low-pass filtered.

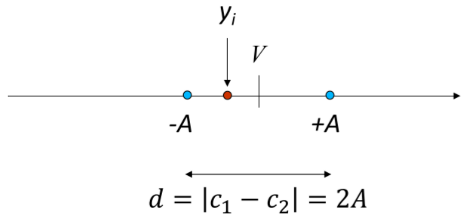
$$\mu_y = \pm A \qquad \sigma^2_y = \frac{k^2 N_0 E}{2} = \frac{N_0}{2T_b}$$



13.1.4 Error in reception

Define the distance between the amplitudes associated to the symbols as $d = |c_2 - c_1|$ and the **decision threshold** $V = c_1 + \frac{d}{2}$. We decide that the closest coefficient to the measured value of y_i has been transmitted. A **transmission error** is made when:

$$y_i > c_1 + \frac{d}{2} = -A + \frac{d}{2} c_1 = -A \text{ has been transmitted}$$
$$y_i < c_2 - \frac{d}{2} = +A - \frac{d}{2} c_2 = +A \text{ has been transmitted}$$



It is possible that the received signal is recognized differently to the transmitted one (*wrong bits*). The possible causes are:

- Thermal noise (transmission medium, transmission and reception devices).
- Interferences of other transmissions through the same medium.
- Electromagnetic disturbances.
- Synchronism loss.
- ...

Conditioned probability (transmitted bit: "0")

The probability error can be computed from the conditioned probability of the Gaussian distribution surpassing the threshold given that an specific symbol has been transmitted.

$$\begin{aligned} P_{e_0} &= P(y_i > V|b_i = 0) = P(y_i > V|a_i = -A) \\ &= \int_V^{+\infty} f_y(y|0)dy = \frac{1}{\sqrt{\pi\frac{N_0}{T_b}}} \int_V^{+\infty} e^{-\frac{(y+A)^2}{N_0/T_b}} dy \end{aligned}$$

Conditioned probability (transmitted bit: "1")

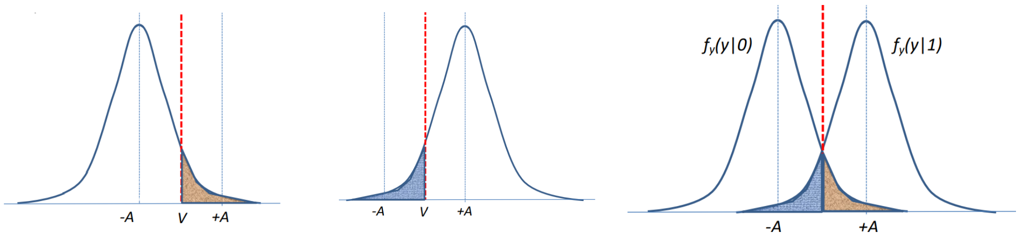
$$\begin{aligned} P_{e_1} &= P(y_i < V|b_i = 1) = P(y_i < V|a_i = +A) \\ &= \int_{-\infty}^{V} f_y(y|1)dy = \frac{1}{\sqrt{\pi\frac{N_0}{T_b}}} \int_{-\infty}^{V} e^{-\frac{(y-A)^2}{N_0/T_b}} dy \end{aligned}$$

Error probability (binary transmission)

The total error probability is very easy to compute using the conditioned probabilities and the occurrence probability of the binary symbols b_i.

$$P_e = P(b_i = 0)P(y_i = 1|b_i = 0) + P(b_i = 1)P(y_i = 0|b_i = 1)$$

Normally $P(0) = P(1) = 1/2$.



(a) Transmitted bit "0". (b) Transmitted bit "1". (c)

Figure 13.2: Error probability in a binary transmission.

To compute the value of the conditioned probability we have to integrate both parts of the Gaussians that go over the threshold V.

$$P_e = \frac{1}{2}\int_V^{+\infty} f_y(y|0)dy + \frac{1}{2}\int_{-\infty}^{V} f_y(y|1)dy$$

To compute these areas, normally the **Q-function** is used, where

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-\frac{u^2}{2}}du$$

If the symbols are equiprobable, the optimal point for the threshold is the one in the middle of the transmitted symbols:

$$V = \frac{c_1 + c_2}{2}$$

In this case the computation of the areas is simplified:

$$\boxed{P_{e_0} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\frac{A}{\sigma_y}\right) = P_{e_1} = P_e}$$

where $E_b = A^2T_b$ and $\sigma_y^2 = \frac{k^2N_0E}{2} = \frac{N_0}{2T_b}$ for a **symmetric binary channel**. So it is enough to compute just one integral, using symmetry.

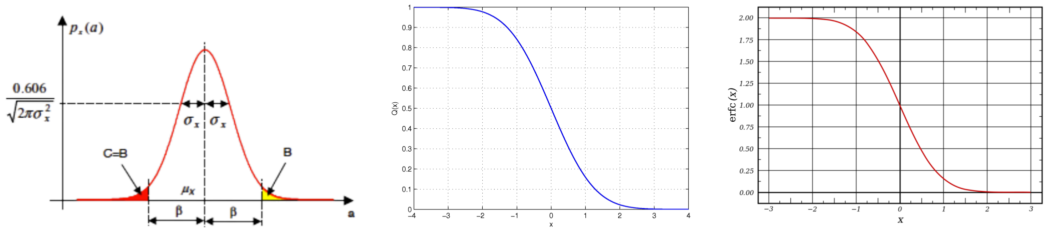
There is another function that we can use to make this calculations: **the complementary error function erfc**:

$$\text{erfc}(z) = 1 - \text{erf}(z) = 1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$$

This allows us to write the error probability as:

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

For binary transmissions, the error probability depends **only** on the rate between the mean energy per bit and the spectral power density of the noise.



(a) $P_x(a)$ (b) $Q(x)$ (c) $\text{erfc}(x)$

$$B = P(x > \mu_x + \beta) = \int_{\mu_x+\beta}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(a-\mu_x)^2}{2\sigma_x^2}\right) da = Q\left(\frac{\beta}{\sigma_x}\right) = \frac{1}{2} erfc\left(\frac{\beta}{\sqrt{2}\sigma_x}\right)$$

(d)

$$\text{per } t > 3 \;\; Q(t) \approx \frac{\exp(-t^2/2)}{\sqrt{2\pi}\, t}$$

$$\text{per } s > 2 \;\; erfc(s) \approx \frac{\exp(-s^2)}{\sqrt{\pi}\, s}$$

(e)

Figure 13.3: Q-function and erfc.

The **error probability** in the case of binary PAM transmission depends **only** on the **distance** between the values associated to the symbols and the power of the noise.

$$P_e = Q\left(\frac{\beta}{\sigma_y}\right) = Q\left(\frac{d/2}{\sigma_y}\right) = Q\left(\sqrt{\frac{d^2}{4N_0B}}\right)$$

A binary PAM transmission system with antipodal symbols (for example $-5, 5$) has the same error probability as a transmission system with unipolar symbols (0,10).

However the **transmitted power** is proportional to $c_1^2 + c_2^2$, so it is double in the case of unipolar symbols. So it possible, it is better to use **antipodal symbols**, as it allows us to spend less power.

In the case of **antipodal** symbols, we find that

$$\frac{d}{2\sigma_y} = \frac{|c_1 - c_2|}{2}\sqrt{\frac{2E_g}{N_0}} = \sqrt{\frac{2c_1^2E_g}{N_0}}$$

in which E_g represents the energy of the waveform $g(t)$ used for the transmission.

In this case the transmission symbols are equivalent to the bits, so

$$E_S = E_b = c_1^2E_g$$

Thus,

$$\boxed{P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2E_S}{N_0}}\right) = P_S}$$

In the case of **unipolar** symbols,

$$c_1 = 0 \qquad \frac{d}{2\sigma_y} = \sqrt{\frac{c_2^2 E_g}{2N_0}}$$

In this case the energy varies depending on the symbol. If they are equiprobable, the mean energy per symbol (and per bit) wile be

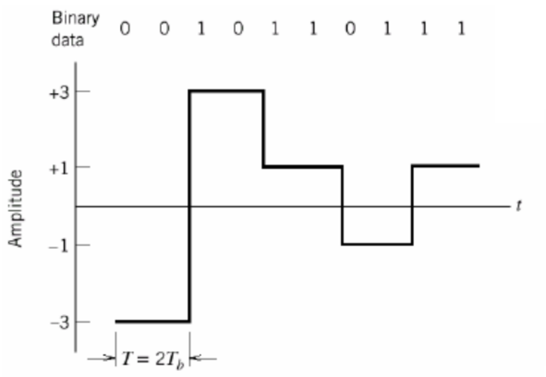
$$E_s = E_b = \frac{c_2^2 E_g}{2}$$

Half of what we have obtained before, *as now half of the time we are in a bit of value 0*. Realize that as smaller this value gets, the error probability increases.

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{E_S}{N_0}}\right) = P_S$$

13.1.5 Multilevel transmission in base-band (M-PAM)

In this case there are M possible a_k corresponding to the same amount of symbols. Each symbol carries $\log_2 M$ bits. If we use 4 levels (2 bits), the symbol time $T = 2T_b$ is twice as long as the bit time.



Dibit	Amplitude
00	–3
01	–1
11	+1
10	+3

Gray code: each symbol differs from the adjacent one only by one bit. It minimizes the error probability, so it is widely used.

Error probability per symbol (M-PAM)

Note: *The symbols are composed by more than one bit, so be careful, the probability per symbol* $\neq$ *the probability per bit.*

In the case of M equidistant levels (each to distance d from the adjacent levels) and equiprobable symbols, the computation of the error probability is simplified.

Take figure 13.4. We have $M - 1$ thresholds. The number of superposed areas is $2(M-1)$. The symbol probability, considering they are equiprobable is $\frac{1}{M}$.

$$P_{e_S} = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{d^2}{4N_0 B}}\right) = 2\left(1 - \frac{1}{M}\right) Q\left(\frac{d}{2\sigma_y}\right)$$

In the general case, we expect a higher error probability than in the binary case.

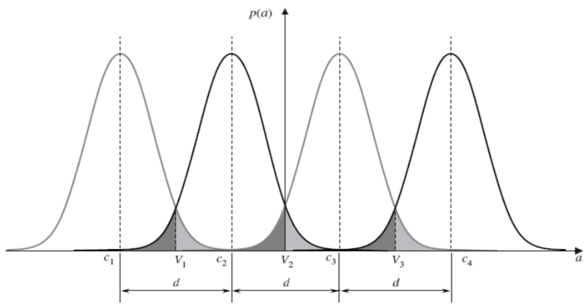


Figure 13.4: $\int_{V_1}^{\infty} f_y(y|c_1)dy = Q\left(\sqrt{\frac{d^2}{4N_0B}}\right)$.

With M equidistant (distance d) and equiprobable levels, it can be shown that

$$\frac{d}{2\sigma_y} = \frac{|c_m - c_{m+1}|}{2}\sqrt{\frac{2E_g}{N_0}} = \sqrt{\frac{(c_m - c_{m+1})^2}{2}\frac{E_g}{N_0}}$$

$$E_S = \frac{1}{M}\sum_{k=1}^{M} E_{S_k} = \frac{d^2E_g}{2M}\sum_{k=1}^{M/2}(2k-1)^2 = \frac{d^2}{2}\frac{M^2-1}{6}E_g$$

This allows us to write:

$$P_{e_S} = 2\left(1 - \frac{1}{M}\right)Q\left(\sqrt{\frac{6}{M^2-1}\frac{E_S}{N_0}}\right)$$

Error probability per bit (M-PAM)

To compare the performance of PAM systems with different amount of symbols, it is convenient to compute the error probability per bit. The relation between the energy per symbol and the energy per bit is very simple:

$$E_S = \log_2 M \, E_b$$

In the case of Gray's encoding, each symbol differs only by one bit from the adjacent symbols. So each time an error occurs, it occurs only by one bit ($1/\log_2 M$). So the error probability is smaller than if we used the natural encoding:

$$\boxed{P_{e_b} \simeq \frac{P_{e_S}}{\log_2 M} = \frac{2}{\log_2 M}\left(1 - \frac{1}{M}\right)Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_b}{N_0}}\right)}$$

Note: The $\simeq$ symbol is used, because we are assuming that the errors that happen are a consequence of missing **only one** bit. We are assuming that errors can **not** take us to symbols further than the adjacent ones. In practice this is quite unusual, but mathematically, the $\simeq$ symbol is more correct than $=$.

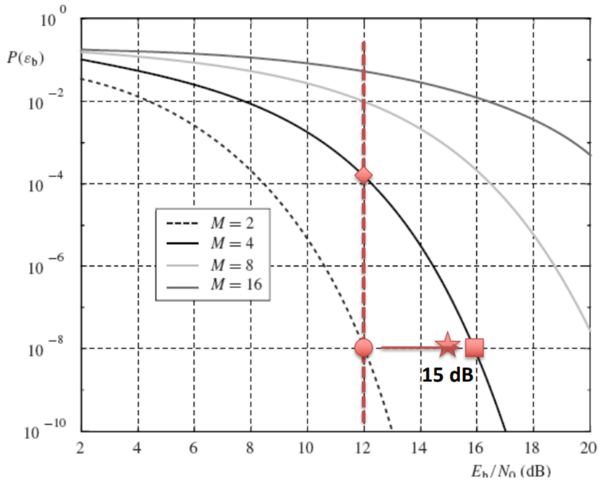


Figure 13.5: $P(\varepsilon_b)$ vs E_b/N_0 (dB).

For equal E_b/N_0, increasing the number of levels implies a much higher error probability.

However, the bit-rate of the transmitted signal is a multiple of the binary case: the frequency band is the same, so the symbols have the same speed, but each symbol has more bits! To obtain the same performance in terms of capacity of the binary case we need a higher E_b/N_0 and a wider band.

If we want to use a multilevel transmission to increase the bit-rate, we also have to increase E_b and the bandwidth (so the power band of the noise also increases). **Penalty**.

13.1.6 Bit error rate

The error probability per bit is normally indicated as *bit error rate* (**BER**). The BER is an indicator that measures the transmission quality of the communication system. It is a synthetic indicator based on other parameters, such as bit-rate and received power.

It tells what to expect when using the communication system, but it does not show the origin of the transmission error.

A transmission system can be considered **error-free** in the case on which the BER is close to $\mathbf{10^{-9} - 10^{-12}}$.

If the error probability needs to be further reduced, error corrector codes can be used (such as *Forward Error Correction* FEC). It does the opposite thing we did when we learned about *source encoding*. It adds some redundancy bits to the signal (*parity*), so that if a limited number of errors happen, they can be corrected: (n, k) code $\Rightarrow$ channel encoding.

$$k \text{ information bits} \mid n-k \text{ partity bits}$$

They are projected to be able to correct up to C errors (corrector power of the code).

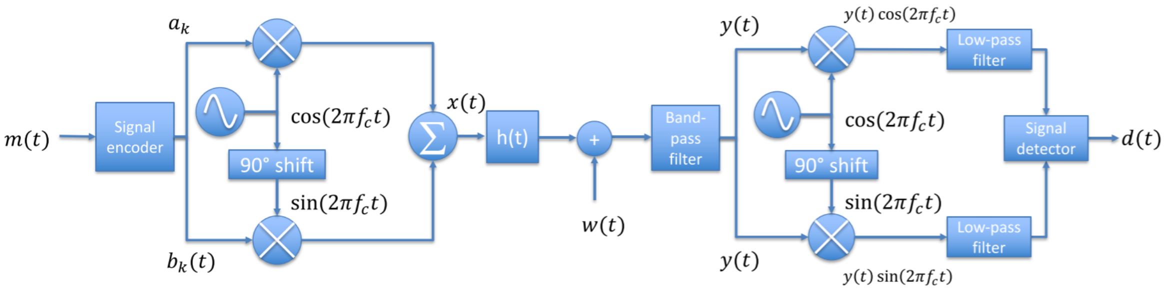
As a consequence of this the bit-rate of the transmitted signal is higher than the one of the original signal.

Example: M-PAM transmission in base-band

- Transmission at $5\,Mb/s$ with **2-PAM** modulation ($M = 2$):
  - Bit time $T_b = 1/5 \cdot 10^6 = 200\,ns$.
  - Minimum band needed: $B_{M=2} = 1/2T_b = 2.5\,MHz$ (roll-off factor $\alpha = 0$).
- Transmission at $5\,Mb/s$ with **4-PAM** modulation ($M = 4$):
  - Symbol time $T_s = 2 \cdot T_b = 400\,ns$.
  - Minimum band needed: $B_{M=4} = 1/2T = 1.25\,MHz$ ($B_{M=4} = B_{M=2}/2$).
  - Needed energy per bit: $5E_g/2$ (4dB more than for the 2-PAM case).
- Transmission at $5\,Mb/s$ with **8-PAM** modulation ($M = 8$):
  - Symbol time $T_s = 3 \cdot T_b = 600\,ns$.
  - Minimum band needed: $B_{M=8} = 1/2T = 833\,kHz$ ($B_{M=8} = B_{M=2}/3$).
  - Needed energy per bit: $7E_g$ (8.5 dB more than for the 2-PAM case, 4.5 dB more than for the 4-PAM case).

13.2 Transmission in translated band

The main difference with respect to base-band transmission is that we can transmit two signals in quadrature. So we have two components, and the noise will distort both of them.



$$\begin{cases} a_k = \mathrm{Re}\{m(t)\} \\ b_k = \mathrm{Im}\{m(t)\} \end{cases}$$

So,

$$x(t) = a_k \cos(2\pi f_c t) + b_k \sin(2\pi f_c t)$$

Therefore,

$$y(t) = x(t) * h(t) * c(t) + w(t) * c(t)$$
$$d(t) \propto a_k + jb_k + n_I + jn_Q$$

The noise acts on both components in an independent way. Be careful: here f_c stands for *carrier*, not *campionamento*.

13.2.1 Effect of the noise: QPSK constellation

We saw that the possible symbols of a transmission in band-pass are organized in **constellations**. We will use these as a geometrical representation of noise.

The effect of the noise on the received symbols: both n_I and n_Q are two Gaussian random variables with null mean and variance σ_w^2.

Therefore, we will have a circular cloud around the nominal values of the symbols. The stars from the figure 13.6a represent a 2D Gaussian distribution.

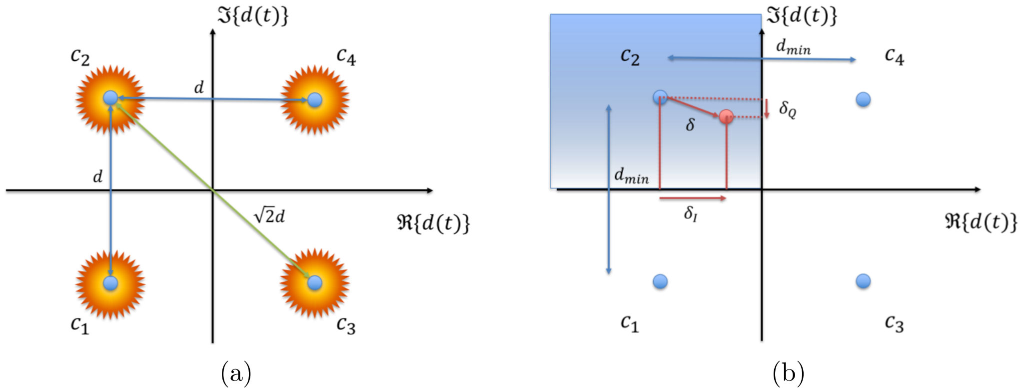


Figure 13.6: QPSK constellation.

As we did in the base band case, it is convenient to compute the distance between the symbols of the constellation, because is useful for finding the error probability of the transmission:

$$d = \sqrt{(a_m - a_k)^2 + (b_m - b_k)^2}$$

Error per symbol

Due to the noise δ, a symbol c_m can be interchanged with an adjacent symbol (fig. 13.6b).

The probability of this change to happen, with the diametrically opposite symbol (c_3 in the figure) is very low and it can be neglected.

In the case of the QPSK modulation, we will have an error when we go out of the colored sector, this is, when

$$\delta_I > \frac{d_{min}}{2} \quad \text{or} \quad \delta_Q < -\frac{d_{min}}{2}$$

Both events can be considered disjunctive, so we find

$$P\left(\delta_I > \frac{d_{min}}{2}\right) = P\left(\delta_Q < -\frac{d_{min}}{2}\right) = Q\left(\frac{d_{min}}{2\sigma_\delta}\right)$$

Therefore the error probability of the symbol c_2 will be:

$$P_{e_{c_2}} = 2Q\left(\frac{d_{min}}{2\sigma_\delta}\right) = P_{e_S}$$

given that the symbols are *equiprobable* and neglecting the probability of going to c_3.

If the symbols are equiprobable, and the distances are always the between adjacent symbols, this probability will be the same for all.

It can be shown[1] that

$$\sigma_\delta = \sqrt{\frac{N_0}{E_g}}$$

The **error probability per symbol** will therefore be:

$$P_{e_s} = 2Q\left(\sqrt{\frac{E_g}{N_0}}\frac{d_{min}}{2}\right) = 2Q\left(\sqrt{\frac{d_{min}^2 E_g}{4N_0}}\right)$$

The probability of an error happening is equal to the *probability of a symbol going to another quadrant* as a consequence of the noise, *multiplied by the number of neighbors.*

Knowing that

$$|c_m|^2 = \frac{d_{min}^2}{2}$$

$$E_s = \frac{1}{2}|c_m|^2 E_g = \log_2 M \, E_b$$

The **error probability per symbol** is:

$$\boxed{P_{e_s} = 2Q\left(\sqrt{\frac{d_{min}^2 E_g}{4N_0}}\right) = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)}$$

[1] The demonstration is in the book.

Error probability per bit

To consider this value is useful as it allows us to compare the error probability of diverse modulations.

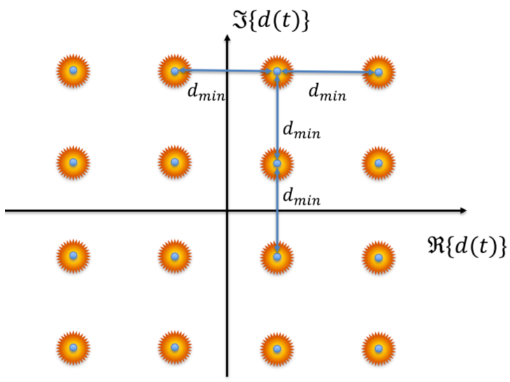
In this case each symbol is formed by 2 bits, and thanks to the Gray's encoding, two adjacent symbols differ only by one bit. The **error probability per bit** in the case of the QPSK modulation will be:

$$\boxed{P_{e_b} = \frac{P_{e_s}}{\log_2 M} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

It is equal to the error probability found for the **2-PAM** with antipodal symbols. For 2-PAM we had a constellation only over the real axis, constituted of 2 points. Now we have 4 points, but in two axis. It is as if we were applying two independent binary antipodal modulations, one to the real part and another one to the imaginary, and that is the reason we find the same probability of the 2-PAM.

Because of this, in general, the modulation called **BPSK** is almost never used, which is constituted of two phase points $(0, \pi)$. Its error probability is the same of the QPSK, but this one doubles the transmission speed or reduces to half the needed bandwidth.

13.2.2 Effect of the noise on M-QAM systems



To compute the error probability for the multilevel systems M-QAM, we work in the same way we have done for the QPSK constellation.

The number of levels is higher, but the constellations are built following the same relations.

This time too, it will be possible to individuate a minimum distance, which on a first approximation it will determine the error probability, neglecting the symbols at higher distance.

Example: 16-QAM constellation

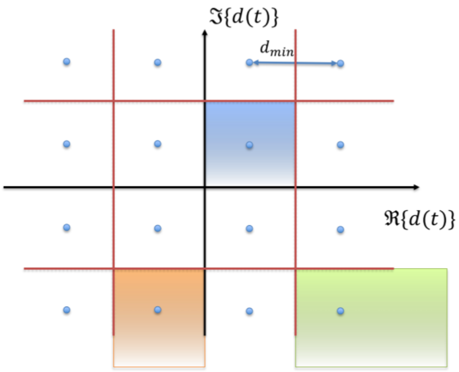


Figure 13.7: 16-QAM modulation.

An error will happen each time the noise moves a symbol to a zone closer to another symbol.

In this case there are three kind of symbols, one in each of the zones colored in the image:

1) 4 internal symbols (*simboli interni*, blue).
2) 4 corner symbols (*simboli di spigolo*, green).
3) 8 border (*simboli di bordo*, orange).

For the **internal symbols**, an error is made every time one of the following happens:

$$\delta_I > \frac{d_{min}}{2},\ \delta_I < -\frac{d_{min}}{2},\ \delta_Q > \frac{d_{min}}{2},\ \delta_Q < -\frac{d_{min}}{2}$$

The four events have probability equal to

$$P\left(\delta_I > \frac{d_{min}}{2}\right) = Q\left(\sqrt{\frac{d_{min}^2 E_g}{4N_0}}\right)$$

The total error probability for these symbols will be the sum of the probabilities of the 4 events:

$$P_{e_{S_1}} \simeq 4P\left(\delta_I > \frac{d_{min}}{2}\right) = 4Q\left(\sqrt{\frac{d_{min}^2 E_g}{4N_0}}\right)$$

For the **border symbols**, an error occurs when one of the following is fulfilled:

$$\delta_I > \frac{d_{min}}{2},\ \delta_I < -\frac{d_{min}}{2},\ \delta_Q > \frac{d_{min}}{2}$$

The 3 events have probabilities equal to:

$$P\left(\delta_I > \frac{d_{min}}{2}\right) = Q\left(\sqrt{\frac{d_{min}^2 E_g}{4N_0}}\right)$$

The total error probability for these symbols will be the sum of the probabilities of the 3 events:

$$P_{e_{S_2}} \simeq 3P\left(\delta_I > \frac{d_{min}}{2}\right) = 3Q\left(\sqrt{\frac{d_{min}^2 E_g}{4N_0}}\right)$$

For the **corner symbols**, an error occurs when one of the following is fulfilled:

$$\delta_I < -\frac{d_{min}}{2},\ \delta_Q > \frac{d_{min}}{2}$$

The 2 events have probabilities equal to:

$$P\left(\delta_I > \frac{d_{min}}{2}\right) = Q\left(\sqrt{\frac{d_{min}^2 E_g}{4N_0}}\right)$$

The total error probability for these symbols will be the sum of the probabilities of the 2 events:

$$P_{e_{S_3}} \simeq 2P\left(\delta_I > \frac{d_{min}}{2}\right) = 2Q\left(\sqrt{\frac{d_{min}^2 E_g}{4N_0}}\right)$$

With 16 equiprobable symbols, the error probability per symbol will be given by the sum of all the error probabilities of each symbol:

$$\boxed{P_{e_S} = \frac{4P_{e_{s1}} + 8P_{e_{s2}} + 4P_{e_{s3}}}{16} \simeq 3Q\left(\sqrt{\frac{d_{min}^2 E_g}{4N_0}}\right)}$$

Where 3 is the average number of neighbors that these points have (check it!).

In the first approximation, the error probability per symbol is determined by *the probability of the noise moving a point to one of the adjacent quadrants, multiplied by the mean number of close symbol.*

Mean energy per symbol in a 16-QAM constellation

To write this probability in terms of the energy used to transmit a symbol, it is considered that:

$$E_S = \frac{1}{2}|c_m|^2 E_g = \log_2 M\, E_b$$

For internal symbols: $|c_m|^2 = \frac{d_{min}^2}{2}$

For border symbols: $|c_m|^2 = 5\frac{d_{min}^2}{2}$

For corner symbols: $|c_m|^2 = 9\frac{d_{min}^2}{2}$

Thus,

$$E_S = \frac{1}{2}\overline{|c_m|^2}E_g = \frac{1}{2}\left[\frac{4 + 8\cdot 5 + 4\cdot 9}{16}\frac{d_{min}^2}{4}\right]E_g = 5\frac{d_{min}^2}{4}E_g$$

13.2.3 Error probability per symbol of a M-QAM constellation

In the case of 16-QAM we have just considered, the error probability per symbol in terms of the mean energy per symbol is:

$$P_{e_S} \simeq 3Q\left(\sqrt{\frac{d_{min}^2 E_g}{4N_0}}\right) = 3Q\left(\sqrt{\frac{E_s}{5N_0}}\right)$$

In general, it can be shown that the expression of the **error probability per symbol** for a number of levels equal to M is

$$\boxed{P_{e_S} = 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_S}{(M-1)N_0}}\right)}$$

13.2.4 Error probability per bit of a M-QAM constellation

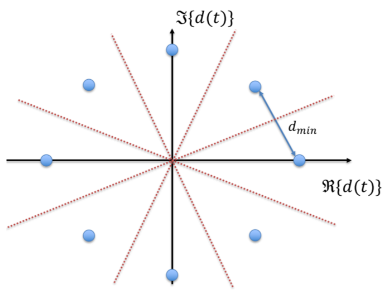
Considering that each symbol represents $\log_2 M$ bits and that we are using Gray's encoding, we find that:

$$\boxed{P_{e_b} = \frac{4}{\log_2 M}\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\log_2 M E_b}{(M-1)N_0}}\right)}$$

Remember that the value of Q decreases as its argument gets bigger. Increasing M, the argument gets smaller and the error probability bigger. To compensate this, we can increase E_b or decrease N_0, tolerate less noise.

In the analog formula for M-PAM, we had $1/(M-1)^2$ inside, so the decrease of the argument was a lot faster (no more than 8 levels). With M-QAM the situation is a bit better (1024 levels can be used, but normally 16-QAM is used).

13.2.5 Error probability of M-PSK constellations



It can be found in an analogous way to what we have done for the **M-PSK modulation**.

The error probability **per symbol** and **per bit** is equal to:

$$P_{e_S} = 2Q\left(\sqrt{\frac{2E_S}{N_0}\sin^2\frac{\pi}{M}}\right)$$

$$P_{e_b} = \frac{2}{\log_2 M}Q\left(\sqrt{2\log_2 M\frac{E_b}{N_0}\sin^2\frac{\pi}{M}}\right)$$

PSK is less convenient than the QAM of the same number of symbols, so it is not so widely used.

Error probability per bit for M-QAM

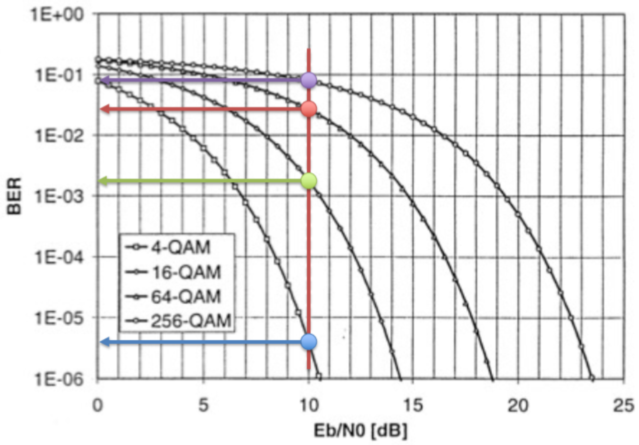


Figure 13.8: Choose a constant E_b/N_0 (the dynamics of the symbols do not change), and see how increasing the number of bits increases the error probability per bit (BER). QPSK≡4-QAM.

The **highest error probability** we can have is 50%, which is the **worst case**, we do not know anything. The lines in the graph tend to this value. Do **not** say that 100% is the highest error probability: this means we are inverting the received bits/signal and have zero error. The worst situation, the maximum error probability is indeed 50%.

In general, using also other tools and special encoding, the error is around 10^{-3} (see figure 13.9).

13.2.6 Characteristics of the BER for different constellations

In general, given a specific E_b/N_0 we find that:

1) BER(M-QAM)$\leq$ BER(M-PSK)$\leq$ BER(M-PAM). The constellation with the lowest error probability is always the M-QAM. Then the M-PSK and finally the

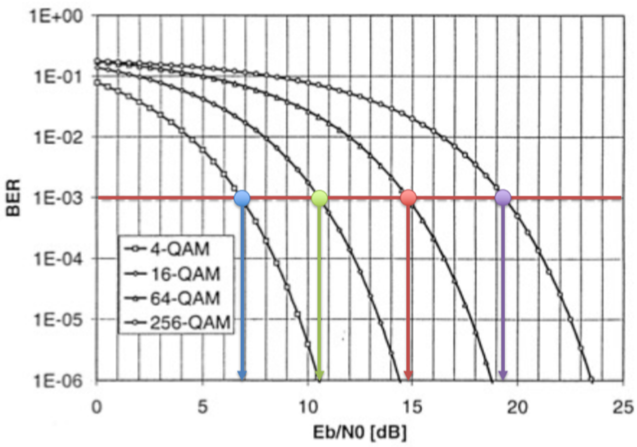


Figure 13.9: Choose a constant error probability per bit (BER) and see how to increase the number of bits an increasing E_b/N_0 (dynamics of the symbols) is needed. We need a bigger *space* to put all these symbols. Even if we increase the number of symbols, if the d_{min} between them is kept fixed, the error probability will remain unchanged.

M-PAM (this one in base-band). The performance in base band is worse with respect to translated band, as we can check on the formulas we have obtained.

2) BER(2-PAM)= BER(2-PSK)=BER(4-PSK)= BER(4-QAM). Where
   - 2-PAM: binary antipodal in base band.
   - 2-PSK: two phases (0,π) in translated band. Equal constellation of the 2-PAM.
   - 4-PSK: four phases in translated band.
   - 4-QAM: quadrature amplitude modulation, equal to 4-PSK.

   When possible, use the modulation with 4 symbols instead of the one with 2 when we are in translated band, because the performance is the same but *with the double of symbols*.
3) For equal constellation families, when increasing M a higher BER is found (more errors) for a fixed E_b/N_0.

13.2.7 How to decide the constellation to use

BER: Bit Error Probability.

To determine the frequency band (base/translated) to use the transfer function of the channel must be analyzed: pass-band of the channel.

Checking the available bandwidth and the transmission-bit rate, the needed number of levels is computed.

The specific modulation topology is determined depending on the complexity and the expected cost of the system.

The theoretical BER is computed to know the power that the transmitter needs to have to reach the target BER, needed to be able to apply an appropriate correction code of the errors and obtain an *error free* transmission.

If it is not possible to reach the needed BER (as a consequence of a too high power, too much noise, non adequate complexity of the system...) we will search a compromise between the obtainable performance and the maximum transmission speed allowed.

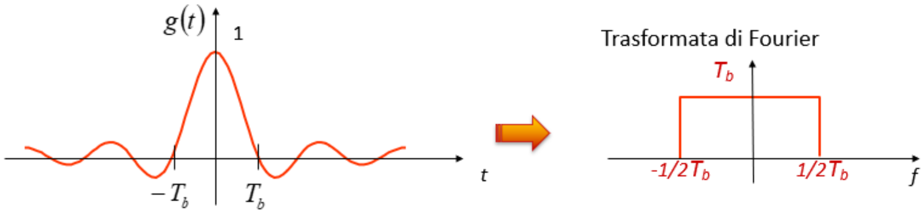
A | ISI & filtro adattato - attachment

A.1 Shape of the ideal impulse

If $H(f)$ and $C(f)$ would not add any distortion to the signal,

$$p(t) = g(t) = \frac{\sin \pi \frac{t}{T_b}}{\pi \frac{t}{T_b}} \quad \Rightarrow \quad G(f) = T_b\, rect(T_b f)$$

If the low-pass channel has unilateral bandwidth B, the shortest bit time T_b that can



be used is equal to $1/2B$, and therefore the **maximum bit-rate** is equal to the double of the bandwidth $(2B)$,

$$\frac{1}{2T_b} = B \quad \Rightarrow \quad \text{bit rate} = \frac{1}{T_b} = 2B$$

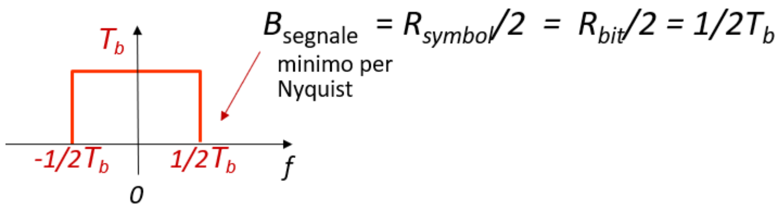
A.2 Low-pass transmission (base band)

A.2.1 Ideal case $(\alpha = 0)$

Binary format transmission: $R_{\textbf{bit}} = R_{\textbf{symbol}} = 1/T_b$

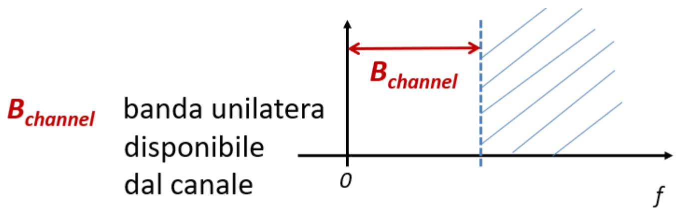
The minimum bandwidth of the signal that fulfills Nyquist's condition, $B_{\text{signal,min Nyq}}$:

$$B_{\text{signal,min Nyq}} = R_{\text{symb}}/2 = R_{\text{bit}}/2 = 1/2T_b$$



We need that:

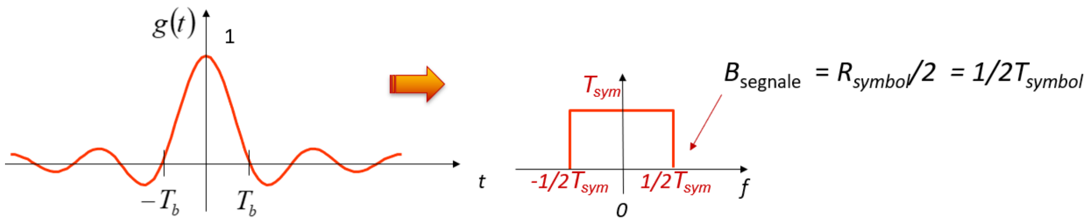
$$B_{\text{signal,min Nyq}} = 1/2T_b \leq B_{\text{channel}} \qquad R_b \leq 2B_{\text{channel}}$$



$$\frac{1}{2T_b} = B_{\text{ch}} \quad \Rightarrow \quad \text{bit rate} = \frac{1}{T_b} = 2B_{\text{ch}}$$

If the low-pass channel has unilateral bandwidth B_{channel}, the shortest **bit time** T_b that can be used is equal to $1/2B_{\text{ch}}$ and therefore the **maximum bit-rate** is equal to the double of the bandwidth of the channel ($2B_{channel}$).

2^M **level format transmission:** $R_{\textbf{symb}} = R_{\textbf{bit}}/M = 1/T_{\textbf{symb}}$



In this case:

$$B_{\text{signal}} = R_{\text{symb}}/2 = 1/2T_{\text{symb}}$$
$$B_{\text{signal}} = 1/2T_{\text{symb}} \leq B_{\text{channel}}$$

Where B_{channel} still is the available unilateral bandwidth of the channel.

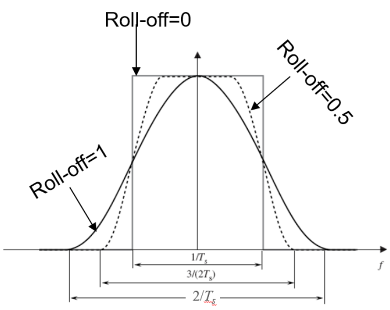
$$\frac{1}{2T_{\text{symb}}} = B_{\text{ch}} \quad \Rightarrow \quad \text{symbol rate} = \frac{1}{T_{\text{symb}}} = 2B_{\text{ch}} \qquad R_{\text{symb}} \leq 2B_{\text{ch}}$$

If the low-pass channel has unilateral bandwidth B_{channel}, the shortest **symbol time** T_{symb} that can be used is equal to $1/2B_{\text{ch}}$ and therefore the **maximum symbol-rate** is equal to the double of the bandwidth of the channel ($2B_{\text{channel}}$).

A.2.2 Non-deal case ($\alpha > 0$)

Binary format transmission: $R_{\textbf{bit}} = R_{\textbf{symbol}} = 1/T_b$

$$B_{\text{signal}} = (1+\alpha)\frac{R_{\text{symb}}}{2} = (1+\alpha)\frac{R_{\text{bit}}}{2} = \frac{(1+\alpha)}{2T_b}$$
$$B_{\text{signal}} = \frac{(1+\alpha)}{2T_b} \leq B_{\text{channel}}$$
$$\frac{1+\alpha}{2T_b} = B_{\text{ch}} \quad \Rightarrow \quad B_{\text{ch}} \leq R_b \leq 2B_{\text{ch}}$$



2^M level format transmission: $R_{\textbf{symb}} = R_{\textbf{bit}}/M = 1/T_{\textbf{symb}}$

$$B_{\text{signal}} = (1+\alpha)\frac{R_{\text{symb}}}{2} = \frac{(1+\alpha)}{2T_{\text{symbol}}}$$

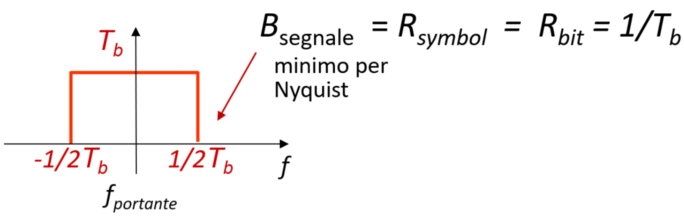
$$B_{\text{signal}} = \frac{(1+\alpha)}{2T_{\text{symbol}}} \leq B_{\text{channel}}$$

$$\frac{1+\alpha}{2T_{\text{symbol}}} = B_{\text{ch}} \quad \Rightarrow \quad B_{\text{ch}} \leq R_{\text{symb}} \leq 2B_{\text{ch}}$$

A.3 Pass-band transmission

A.3.1 Ideal case ($\alpha = 0$)

Binary format transmission: $R_{\textbf{bit}} = R_{\textbf{symbol}} = 1/T_b$



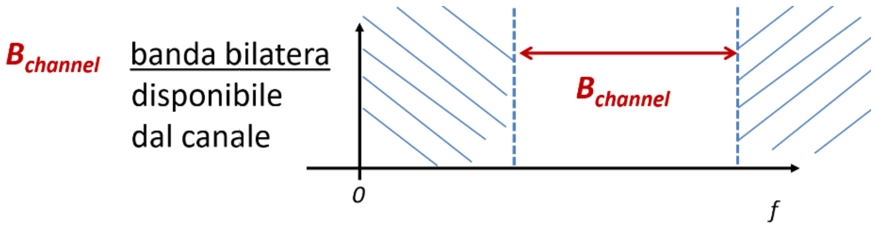
$$B_{\text{signal,min Nyq}} = R_{\text{symb}} = R_{\text{bit}} = 1/T_b$$

$$B_{\text{signal,min Nyq}} = 1/T_b \leq B_{\text{channel}}$$

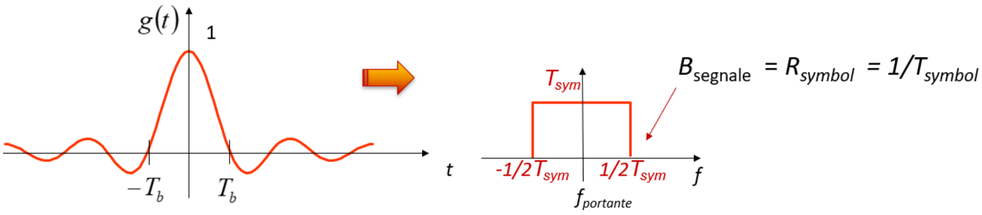
where B_{channel} is the available **bilateral** bandwidth of the channel.

$$\frac{1}{T_b} = B_{\text{ch}} \quad \Rightarrow \quad R_{\text{bit}} = B_{\text{channel}}$$

If the band-pass channel has bilateral bandwidth B_{channel}, the lowest **bit time** T_b that can be used is equal to $1/B_{channel}$ and therefore the **maximum bit-rate** is equal to the bandwidth of the channel B_{channel}.



2^M **level format transmission:** $R_{\text{symb}} = R_{\text{bit}}/M = 1/T_{\text{symb}}$



$$B_{\text{signal}} = R_{\text{symb}} = 1/T_{\text{symb}}$$

$$B_{\text{signal}} = 1/T_{\text{symb}} \leq B_{\text{channel}}$$

$$\frac{1}{T_{\text{symb}}} = B_{\text{ch}} \quad \Rightarrow R_{\text{symb}} = B_{\text{channel}}$$

If the band-pass channel has bilateral bandwidth B_{channel}, the shortest **symbol time** T_{symb} that can be used is equal to $1/B_{\text{channel}}$ and therefore the **maximum symbol-rate** is equal to the bandwidth of the channel B_{channel}.

A.3.2 Non-ideal case ($\alpha > 0$)

Binary format transmission: $R_{\text{bit}} = R_{\text{symbol}} = 1/T_b$

$$B_{\text{signal}} = (1+\alpha)R_{\text{symb}} = (1+\alpha)R_{\text{bit}} = \frac{(1+\alpha)}{T_b}$$

$$B_{\text{signal}} = \frac{(1+\alpha)}{T_b} \leq B_{\text{channel}}$$

$$\frac{1+\alpha}{T_b} = B_{\text{ch}} \quad \Rightarrow \quad B_{\text{ch}}/2 \leq R_b \leq B_{\text{ch}}$$

2^M **level format transmission:** $R_{\text{symb}} = R_{\text{bit}}/M = 1/T_{\text{symb}}$

$$B_{\text{signal}} = (1+\alpha)R_{\text{symb}} = \frac{(1+\alpha)}{T_{\text{symb}}}$$

$$B_{\text{signal}} = \frac{(1+\alpha)}{T_{\text{symb}}} \leq B_{\text{channel}}$$

$$\frac{1+\alpha}{T_{\text{symb}}} = B_{\text{ch}} \quad \Rightarrow \quad B_{\text{ch}}/2 \leq R_{\text{symb}} \leq B_{\text{ch}}$$

B | Graph of the Q-function

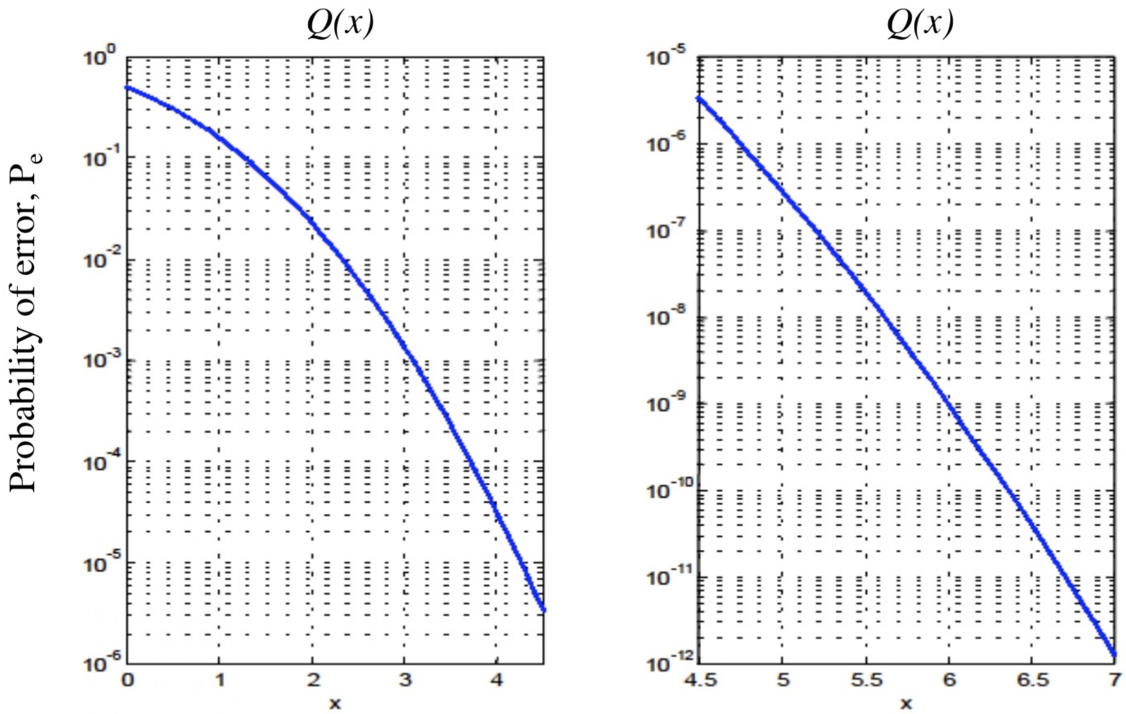


Figure B.1: Average probability of symbol error P_e